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BROAD-BASED HIGH-FREQUENCY JET ENGINE CONTROL

Final
~~Progress~~ Report
~~January 1995~~ - July 1995
Oct 1993

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Carlos Rivera

Abstract

The focus of the effort being carried out under the AFSOR grant is to fully characterize the behavior of rotating stall phenomenon in turbomachinery systems and to assess effectiveness of nonlinear control strategies using both CFD simulations and experiments on the axial compressor facility in the LICCHUS lab. Towards accomplishing this objective, a CFD model has been developed and it has been validated using experimental data available in the literature. The axial compressor facility in the LICCHUS lab will be used for correlation with CFD simulations and for stall controller studies.

1 Introduction

A renewed interest in modeling and simulation of aircraft engine components has emerged during the past twenty years. One motivation for such interest is the prediction of unsteady stall phenomena in compressors and its relation to fluid dynamic stability. Modeling of complex turbomachinery systems has grown to embrace the field of modern control theory, in search for mechanisms of instability suppression.

The pursue of simplified models stems from the motivation of reducing the dimensionality and associated complexity of complete aerodynamic simulations. While these efforts offer the advantage of computational economy, the simplifications made result in the residualization of physical system dynamics which might be of interest, and whose detailed study could reveal finer details of instability inception and development. However, models that reduce to systems of ordinary differential equations possess and additional advantage. The impressive advances in the numerical analysis, simulation, and control of complex, ODE–based nonlinear systems have allowed the development of control strategies for turbomachinery flow instabilities based on such models. In this manner, the reduction or elimination of the design stall margin through active control has become a major objective for addressing instability phenomena in turbomachinery.

In principle, the practical validation and implementation of these control strategies will enable the design and manufacture of more robust, high–performance machinery. However, as pointed out earlier, most existing models lack quantitative description capabilities for most physical systems. When such models are employed, the physical system under consideration must be tailored to the model by means of system identification tools and a certain degree of empiricism, given the standing ideal–behavior assumptions present in these models. Such detailed numerical analysis, transient parameter adjustment, and map identification are required when the control law to be implemented in a physical system is derived from a mathematical model.

While no simplified model can adequately capture the behavior of a physical system, it is believed that the element of empiricism can be removed from a model formulation by studying detailed physical aspects of the modeled system. For example, in a post–stall model validation program carried out at Georgia Tech it was found that the addition of input terms to mimic the behavior of rotor disturbances results in some improvements in the dynamic response of the modeled system. Simplified models available in the present day literature do not address the disturbances created by the rotors and blade row interference effects. While these issues can be studied through experiment, the advances in instrumentation and measurement in turbomachinery have yet to insure an increased understanding of the flow pattern. On the other hand, it has been established in recent times that flow field information obtained from the numerical solution of the fundamental equations of fluid motion in their different levels of simplification has become a profitable means of obtaining and analyzing data for complex turbomachinery flows. Given that the behaviors which can be analyzed by detailed flow simulations are quite often those which have a large impact on the response of these complex flow systems, one can identify an incentive for considering such calculations, as they provide information not completely conveyed by simplified models.

In this spirit, the work that has been performed under the grant will take on the renewed objective of demonstrating the feasibility of simulating stall phenomena in turbomachinery based on the numerical solution of the Navier–Stokes equations. The simulation of stalled flows in turbomachinery is a virgin area of research; it is expected that this work will be one of the first few in its class. In addition to these efforts, the numerical calculations will be used in conjunction with the one of the simplified–model–based instability control concepts presented in the growing literature of stall control in an attempt to quantify the degree of success of such strategies.

The major contribution of this work to the existent literature is expected to be the practical demonstration of the possibility of increasing current knowledge of rotating stall instabilities by the application of well established tools in Computational Fluid Dynamics (CFD). It is also expected that exploring and analyzing simple simulation results will generate ideas for better guidance on post–stall modeling of rotating machinery.

2 Progress in 1995

During the past year, validation of a simplified post–stall model against experimental data from an axial flow compressor rig at Georgia Tech LICCHUS[1] has been carried out. The focus has been the validation of nonlinear behaviors associated with bifurcations in the system dynamics, as captured by the model. Steady–state validation has been carried out through nonlinear programming techniques. These optimal results work quite well but good matching over some flow/pressure regimes is often hard to obtain; however, the technique can be used directly in a robust control design methodology, in order to account for uncertainty in the model. In addition, the control approach presented in the grant–supported publications "*A Simplified Approach for Control of Rotating Stall, Part I: Theoretical Development and Part II: Experimental Results*" (AIAA Journal, (11)6, 1195 – 1223, 1995) has been tested in the model, to complement the experimental and analytical results presented in those papers. The results obtained for the validated model under controlled operation (system gains and parameters) will be used, along with the numerical simulation, to establish the effect of this particular controller.

[1] Acronym for Laboratory for Identification and Control of Complex, Highly Uncertain Systems.

In order to perform the main task of simulating stalled flows in a compressor rotor, a two–dimensional multiblock Navier–Stokes based flow solver called MFOIL and developed by Prof. L.N. Sankar at Georgia Tech, has been modified for the application of solving steady and unsteady flows in cascades. During the past year and after approximately nine months of work, confidence in the code has been established by comparing its output to experimental data, both in steady and unsteady flow scenarios. The numerical approach is based on the well–known Alternating Direction Implicit (ADI) scheme employing an Approximate Factorization (AF) on the linearization of the equations.

Typical simulation results are shown in Figures 1 and 2. Figure 1 shows a comparison of the blade loads on a NACA 65-410 profile cascade at zero angle of attack ($\alpha_1 = 0$), 30 degrees flow off the axial direction ($\beta_1 = 30^o$), cascade solidity, σ, of 1.25, free–stream Mach $M_\infty = 0.085$, and Reynolds number $Re = 245,000$. In these results, loading is reported in terms of total–to–static pressure difference,

$$S_{surface} = \frac{p_0 - p_{surface}}{\frac{1}{2}\rho_\infty V_\infty^2}$$

where p_0 is the stagnation pressure of the upstream flow. For incompressible flow, S is simply $1 - C_p$, where C_p is the standard static–to–static pressure difference coefficient. The experimental data comes from the NACA Report no. 1368 published in 1958 by J. C. Emery, L. J. Herrig, J. R. Erwin and R. Felix under the title "*Systematic Two-Dimensional Cascade Tests of NACA 65–Series Compressor Blades at Low Speeds*". Other cases in this report compared well with the output of the program. Minor discrepancies can be attributed to the application of boundary conditions for the curvilinear grid employed and perhaps an unaccounted small bending of the blades under load in the experiment.

Figure 2 shows a comparison of the amplitude, $|C_{p_a}|$ and phase relative to input of the lifting pressure coefficient $C_{p_a} = C_{p_{upper}} - C_{p_{lower}}$ of a flat plate cascade in which the blades undergo simple–harmonic pitching about their leading edges. The phase angle between the motion of neighboring blades, otherwise known as the interblade phase angle (IBPA), is zero in this case, as well as the mean angle of attack and blade stagger angle, $\gamma = \beta_1 - \alpha_1$. The free stream Mach number is 0.5 and the pitch angle amplitude, α_p is 0.01 degrees. The reduced frequency of oscillation based on semichord is $k = \omega c/(2V_\infty) = 0.5$, where ω, c and V_∞ are the circular oscillation frequency, blade chord length, and free–stream speed, respectively. The results are compared to the output of LINSUB, a numerical program written by D. S. Whitehead for the calculation of linearized unsteady subsonic flows in cascades and presented in volume 1 of the AGARD Manual on Aeroelasticity in Axial Flow Turbomachines (AGARDograph no. 298, 1987). The output of MFOIL varies considerably more than that of LINSUB over the chord length. The discrepancy is a result of the grid resolution, coupled with a time–varying spatial accuracy at the surface as a result of the currently–implemented grid deformation technique for the blade motion. Otherwise, the MFOIL output is seen in good qualitative agreement with that of LINSUB. Although this issue could be addressed by implementing a grid–deformation method which does not allow the grid near the solid surfaces to change in time, the exploratory research to be carried does not call for this type of blade surface motions. Such motions have only been coded to validate the cascade–modified MFOIL program in unsteady flow cases.

3 Plan for 1996

The primary objective for this research period will be the successful simulation of rotating stall in an isolated rotor based on the numerical solutions obtained through MFOIL. To that end, it is expected that the first five months of the year will allow a study of the parameters that will be required to obtain a propagating stall scenario in such rotor.

Once the the occurrence of rotating stall is established, the issue of comparing simulation outputs with simplified model outputs will be undertaken. According to simplified models, it is necessary to account for the fact that nonlinearity has a strong effect on the behavior of the mathematical formulations which model these systems. In particular, the presence of finite domains of attraction of certain operating conditions will be investigated, given that this concept has allowed the development of a stall controller by O. O. Badmus at Georgia Tech and documented in his Doctoral Thesis. This controller will be implemented in the numerical code to assess the effectiveness of the control strategy and to point out the feasibility of improving modeling through CFD simulations. This part of the work is expected to take up the second part of the year and into the year 1997.

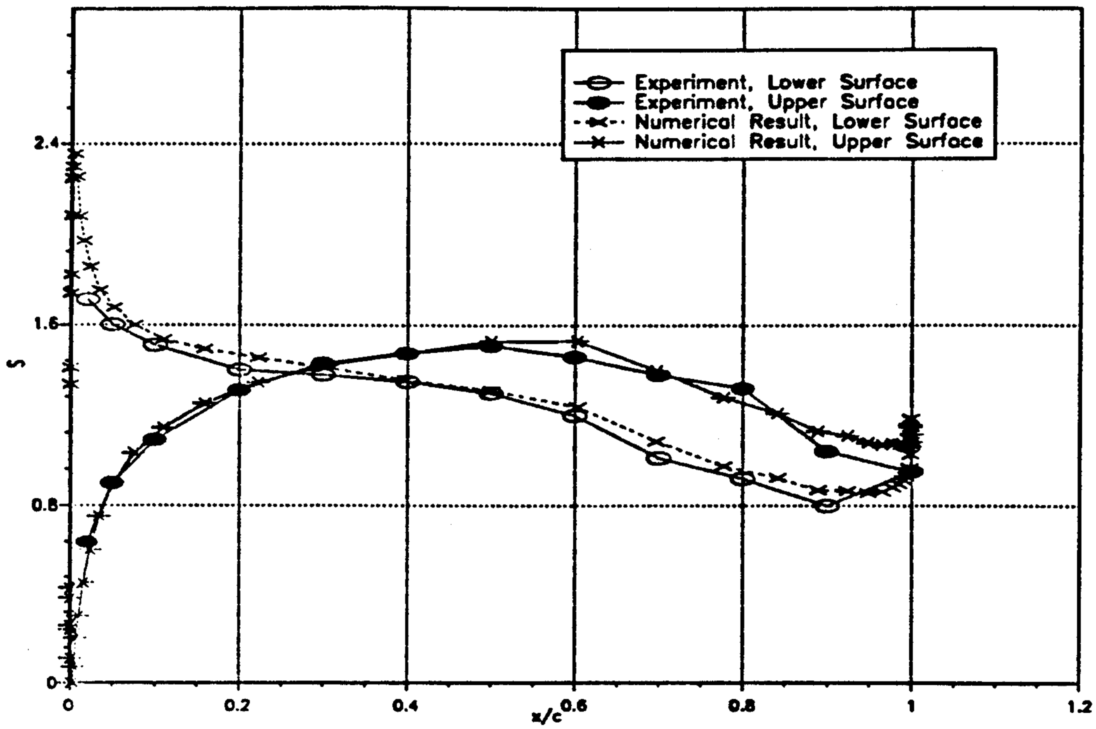

Figure 1: Comparison of Experimental and Numerical Results for Cascade Loading on NACA 65-410 profile, $M_\infty = 0.085, Re = 245,000,\ \alpha_1 = 0,\ \beta_1 = 30°,\ \sigma = 1.25$.

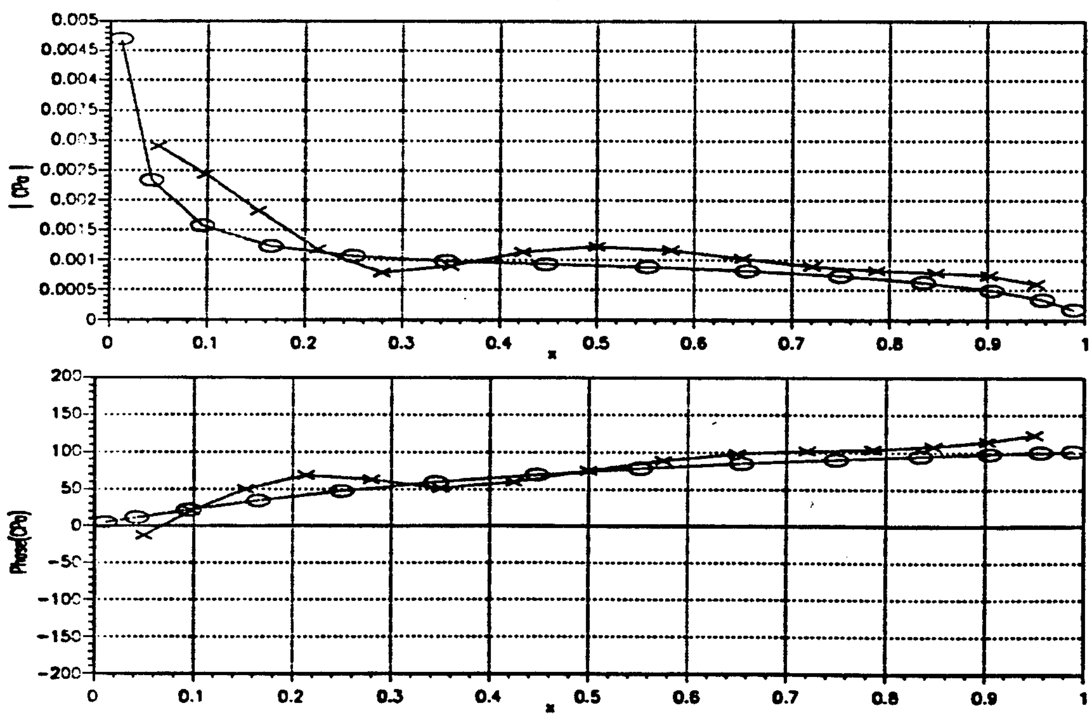

Figure 2: Comparison of LINSUB ("O") and Numerical Results ("X") for Lifting Pressure Coefficient C_{p_a} on Flat Plate Cascade Oscillating About Leading Edge, $k = 0.5,\ M_\infty = 0.5,\ IBPA = 0$.

"DRAFT"

FINAL TECHNICAL REPORT

FROM SUBPROJECT

BROAD-BASED HIGH-FREQUENCY JET ENGINE CONTROL

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October 1, 1993 - September 30, 1994

ABSTRACT

For the control of combustion instability in a ramjet engine by fuel injection, a mathematical model of Fung, Yang, Sinha and Menon reduces to a forced wave equation with delayed boundary control. The first and only step in the analysis that was completed dealt with the unforced problem which exhibits instabilities even for small delays. The main contribution of our analysis was to relate these instabilities to the instabilities in difference equations, a problem that is better understood than the wave equation.

Delayed Feedback Control of Hyperbolic PDE and Combustion Instabilities

The Physical Problem. Combustion instability in a ramjet engine involves nonlinear interaction among acoustic waves, vortex motion and unsteady heat release. The instability is often observed as a large amplitude pressure oscillation in the low frequency range. At some critical limit of the amplitude, there can be structural damage due to fatigue or there can be engine unstart. The latter occurs when the shock in the inlet duct is expelled to form a bow shock ahead of the inlet and is one of the most serious problems in developing an operational ramjet engine.

The results of experimental ([1], [2]) and numerical studies ([3]) indicate that the pressure fluctations in the combustor grow to large amplitude low frequency oscillations when the unsteady heat release in the combustor is in phase with the pressure fluctuation. These studies showed also that a large scale vortex/flame structure propagates through the combustor at the same frequency. This suggests some nonlinear coupling between the shear flow, the pressure oscillation and the unsteady heat release.

There have been many attempts to suppress these instabilities using both passive controls (geometric modifications) and active controls (acoustic feedback control and secondary fuel injection control) (see [4] for a survey). Acoustic controllers appear to be impractical due to the hostile environment in the combustor and the large power requirements. Active control techniques using secondary fuel injection have been shown to be effective in certain situations and also have the possibility of increasing the net thrust of the engine (see [5]), and therefore seem to be a promising approach to the control of full scale ramjet engines.

Practical application of secondary fuel injection control in a full scale combustor has not been demonstrated. Part of the difficulty arises from the fact that, in a high flow rate combustor, the combustion is irregular and multiple frequencies are amplified. In this case, the dominant instability mode can change with the operating conditions. Even when control is applied to suppress the dominant mode, additional new frequencies can be excited and become unstable (see [6],[7],[8]).

A Mathematical Model. Our study is concerned with an active control based on fuel injection based on a model developed in [9], [10], [11]. This approach is based on a wave equation which describes the dynamic behavior of nonlinear oscillations with distributed feedback actions. The resulting boundary value problem is a forced linear wave with the forces including all influences of acoustic motion, mean flow and combustion response under conditions without external forcing together with the effects of the control inputs, one of which is a source term.

A closed loop control system is obtained by relating the source term to the mass flow rate of the injected fuel by means of a time lag theory developed in [12]. This term is proportional to the rate of change of the injection rate of the control fuel at a delayed time and a spatial distribution function characterizing the fraction of the fuel element burned at a fixed position with a time delay with respect to the moment of injection.

The system is implemented by first determining the state of the acoustic field by monitoring the instantaneous pressure signature at a fixed location, for example, at the wall near the downstream diffuser ([13]). This can be interpreted as a point sensor, located at a fixed position with a certain amplication factor c. Following the idea in [12], a Proportional-plus-Integral (PI) controller is then introduced to modulate the mass injection rate of the secondary fuel. Since the time derivative of the fuel injection rate exerts direct influence on the acoustic field, the PI control law is equivalent to a Proportional-plus-Derivative (PD) control law. Therefore, the acoustic pressure input produced by the combustion of the injected fuel is given as a linear function of the error and its rate of change at a delayed time, with proportionality constants K_P and K_D. The time delay between the sensor output and fuel injecton accounts for the actual time required for data acquisition, signal processing and dynamic response of the fuel injection mechanism.

The parameters K_P and K_D are control parameters. In addition to these parameters, there is the freedom in the selection of the time delays and the spatial distribution of the external forcing associated with the burning of the injected fuel.

In [11], the authors approximated the above problem by using a finite number of modes and gave a procedure to relate the control parameters K_P, K_D and the time delays τ_k' to the elimination of instabilities. Numerical results also were obtained which showed that the reduction of the amplitude of the lowest mode was approximately the same as obtained by the experimental results in [6]. On the other hand, it was also noted numerically that another low frequency was excited. They were unable to explain the source of this mode due to the fact that the simulation was not carried out long enough to obtain sufficient data to spectrally resolve this low frequency.

Even though the above model applies only to acoustic effects (ignoring all convective components), there are still many unanswered questions which appear to be fundamental. This is particularly true with respect to the manner in which stabilization of some modes seems to lead to instability of other modes.

The effects of delays in boundary control. The above model in combustion reduces to a discussion of the dynamics of a wave equation when it is subjected to a delayed control at some points on the boundary. It has been known for some time that the introduction of the delay can lead to unexpected behavior in the dynamics. For example, for the linear unforced wave equation for which instantaneous boundary control yields exponential stability, it is known that arbitrarily small time delays can destroy stability (see [14], [15], [16]).

The fact that phenomenon of this type could occur had been observed long before these particular control problems were considered (see [17], [18], [19], [20], [21], [22]). Although these earlier investigations were concerned mainly with neutral delay differential equations, a simple change of coordinates shows that the above hyperbolic PDE is equivalent to a neutral delay differential equation. The instabilities are a consequence of the dynamics associated with a difference equation arising from the neutral delay differential equation.

The main accomplishment so far in this research has been to make this identification clear. The problem is still under investigation.

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Effects of delays on dynamics

by

J.K. Hale

CDSNS94-200

Effects of Delays on Dynamics

by

Jack K. Hale*

Abstract

Part one of these lectures is devoted to two fixed point theorems motivated by dynamics in delay equations: an aympototic fixed point theorem which can be applied to determine ω-periodic solutions of an ω-periodically forced equation and an ejective fixed point theorem which can be applied to the determination of nontrivial periodic solutions of autonomous equations. Part two is devoted to large delays, Hopf bifurcations and fixed points of maps. Part three shows that small delays can destroy stability properties in delay differential equations as well as boundary control of partial differential equatons.

Key Words: Delay equations, dynamical systems, asymptotic fixed point theorems, ejective fixed point theorems, periodic solutions, stability, singular perturbations, boundary control.

Part 1. Fixed Point Theorems Motivated by Dynamics

1. Asymptotic Fixed Point Theorems. Let us begin by considering the ordinary differential equation

$$\dot{x} = f(t, x) \tag{1.1}$$

where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function, $f(t,x) = f(t+\omega, x)$ for all t, x. It is of interest for such an equation to know if there exists a periodic solution of the same period as the vector field; namely, an ω-periodic solution. If the equation arises as a model of a physical system, then it is expected that all of the solutions will be defined for $t \geq 0$ and also that they remain bounded. Is this enough to imply that there is an ω-periodic solution? In general, this is not the case and determining conditions under which this is true has led to interesting asymptotic fixed point theorems. In this section, we survey some of the known results on this problem. We point out also the types of conditions that will imply similar results for problems in infinite dimensions and that will have applications to functional differential equations and partial differential equations.

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The determination of ω-periodic solutions of (1.1) is equivalent to finding the fixed points of the Poincaré map T, where $Tx_0 = x(\omega, x_0)$ and $x(t, x_0)$ is the solution of (1.1) satisfying $x(0, x_0) = x_0$. The following result is due to Massera (1950).

Theorem 1.1.

(i) If $n = 1$ and there is a solution of (1.1) bounded on $[0, \infty)$, then there is an ω-periodic solution.

(ii) If $n = 2$, all solutions of (1.1) are defined for $t \geq 0$ and there is a solution of (1.1) bounded on $[0, \infty)$, then there is an ω-periodic solution.

(iii)If $n = 2$, then there is a vector field f such that there is a solution of (1.1) bounded on $[0, \infty)$ and there does not exist an ω-periodic solution.

(iv) If $n = 3$, then there is a vector field f such that all solutions of (1.1) are bounded on $[0, \infty)$ and there does not exist an ω-periodic solution.

The proof of (i) is very easy because the Poincaré map is monotone. In fact, if $x(t, x_0)$ is bounded on $[0, \infty)$, then the sequence $T^k x_0$, $k = 1, 2, \ldots$, is bounded and monotone. Thus, there is a limit x^* and it must be a fixed point of T. The proof of (ii) is more difficult and uses a fixed point theorem of Brouwer which relies heavily upon the fact that the dimension of the space is two. The construction of counterexamples to the result for $n = 2$, $n = 3$ as stated in (iii), (iv) are difficult (for another example for $n = 3$ due to S.-N. Chow, see Yoshizawa (1975)).

In the case where the vector field in (1.1) is affine, it is possible to show that the existence of a bounded solution implies the existence of an ω-periodic-solution (see Yoshizawa (1975), for example). This result also is true for affine mappings T on infinite dimensional spaces. To state the result, we need some additional notation which also will play an important role in our subsequent discussion of nonlinear problems. For any bounded subset B of a Banach space X, the Kuratowski measure of noncompactness $\alpha(B)$ is defined as

$$\alpha(B) = \inf\{d : B \text{ has a finite cover } U \text{ of diameter } < d\}.$$

A map $T : X \to X$ is said to be *condensing* if $\alpha(T(B)) < \alpha(B)$ for any bounded set B in X for which $\alpha(B) > 0$. If, in addition, there exists a constant $k, 0 \leq k < 1$, such that $\alpha(T(B)) \leq k\alpha(B)$ for any bounded set B in X, then T is said to be an *α-contraction.* The following result is essentially due to Chow and Hale (1974). We give the proof since it is very simple and uses the fixed point theorem of Massatt (1980) (which generalizes a result of Darbo (1955) on α-contracting maps) and states that a condensing map of a closed bounded convex subset of a Banach space into itself must have a fixed point.

Theorem 1.2. *If X is a Banach space, $z \in X$ is fixed, L is a linear α-condensing map on X, $Tx = Lx + z$ for $x \in X$ and there is an $x_0 \in X$ such that $\{T^k x_0, k = 0, 1, \ldots\}$ is bounded, then there is a fixed point of T in X.*

Proof. Let D be the convex hull of the set $\{x^n = T^n x_0, n = 1,2,\dots\}$. If $y \in D$, then $y = \Sigma_{i\in J}\beta_i x^i$, where J is a finite subset of the positive integers, $\beta_i \geq 0$, $\Sigma_{i\in J}\beta_i = 1$. Since

$$\begin{aligned} Ty = Ly + z &= L(\Sigma_{i\in J}\beta_i x^i) + (\Sigma_{i\in J}\beta_i)z \\ &= \Sigma_{i\in J}\beta_i(Lx^i + z) = \Sigma_{i\in J}\beta_i Tx^i \in D, \end{aligned}$$

we have $T(\bar{D}) \subset \bar{D}$. Since T is α-condensing, there is a fixed point of T in $\bar{D}$.

Let us return to the nonlinear system (1.1). Levinson (1944), in his study of the periodically forced van der Pol equation, initiated the study of the modern theory of dissipative systems. If T is a continuous map on a Banach space X, we say that T is *point dissipative* if there is a bounded set B in X if, for any $x \in X$, there is an $n_0 = n_0(x, B)$ such that $T^n x \in B$ for $n \geq n_0$. If X is finite dimensional and T is point dissipative, Levinson (1944) showed that the map T has a maximal compact invariant set $\mathcal{A}$ (that is, $T\mathcal{A} = \mathcal{A}$). He then showed that the Poincaré map for the periodically forced van der Pol equation was point dissipative and used the above property of existence of a maximal compact invariant set to show that there is an integer k such that there is a periodic solution of period $k\omega$; that is, a subharmonic of order k. From Theorem 1.1(ii), since $n = 2$, we know that there must be a periodic solution of period ω. Other results allow us to prove the following rather remarkable result.

Theorem 1.3. *If the Poincaré map for (1.1) is point dissipative, then there is a fixed point of T.*

A proof of this result can be supplied (see, for example, Pliss (1966), Yoshizawa (1975)) using the following asymptotic fixed point theorem of Browder (1959):

Theorem 1.4. *Let S and S_1 be open convex subsets of the Banach space X and let S_0 be a closed convex subset of X such that $S_0 \subset S_1 \subset S$. If $T : S \to X$ is continuous, $T(S)$ is contained in a compact set of X and if, for a positive integer m, T^m is defined on S_1 and $\cup_{0\leq j\leq m} T^j S_0 \subset S_1$ and $T^m S_1 \subset S_0$, then T has a fixed point in S_0.*

It was also observed by Gerstein and Krasnoselskii (1968), Horn (1970), Billotti and LaSalle (1971) that Theorem 1.4 could be used to extend Theorem 1.3 to the case where T is completely continuous. Jones (1965) and Yoshizawa (1966) used a similar method to obtain ω-periodic solutions of retarded functional differential equations (RFDE) when the period ω is at least as large as the delay. This condition was imposed to have the Poincaré map completely continuous.

Theorem 1.3, when extended to completely continuous maps on a Banach space, is very useful in the applications to parabolic partial differential equations (parabolic PDE) and to RFDE when the period $\omega \geq r$, where r is the delay. What happens when $\omega < r$? What about other applications that deal with neutral functional differential equations

(NFDE) and damped hyperbolic PDE? Extensions have been made which are sufficiently general to apply to these situations. To describe one of the main results, we need some additional concepts. We say that the continuous map $T : X \to X$ is *compact dissipative* if there is a bounded set B in X such that, for any compact set $K \subset X$, there is an integer $n_0 = n_0(K, B)$ such that $T^n K \subset B$ for $n \geq n_0$. The following result was independently proved by Hale and Lopes (1973), Nussbaum (1972).

Theorem 1.5. *If X is a Banach space and $T : X \to X$ is condensing and compact dissipative, then T has a fixed point.*

A set K is said to *attract points* (*attract compact sets*) (*attract bounded sets*) of a set B if, for any $\epsilon > 0$, and any point $x \in B$ (compact set H in B) (bounded set U in B), there is an integer n_0 such that $T^n x$ $(T^n H)$ $(T^n U)$ belongs to the ϵ-neighborhood of K for $n \geq n_0$. It is known that the conclusion of Theorem 1.5 is still valid if we replace 'compact dissipative' by 'there is a compact set which attracts compact sets of X' (see, for example, Hale and Lopes (1973), Nussbaum (1972), or Hale and Verduyn-Lunel (1993)).

We give a brief indication of the proof of Theorem 1.5 following the ideas in Hale and Lopes (1973) since it can be deduced from the following result of Horn (1970), which is an extension of Browder's Theorem 1.4.

Theorem 1.6. *Let $S_0 \subset S_1 \subset S_2$ be convex subsets of a Banach space X with S_0, S_2 compact and S_1 open in S_2. Let $T : S_2 \to X$ be a continuous mapping such that, for some integer $m > 0$, $T^j S_1 \subset S_2$ for $0 \leq j \leq m-1$ and $T^j S_1 \subset S_0$ for $m \leq j \leq 2m-1$. Then T has a fixed point.*

To use this result, Hale and Lopes (1973) first prove the following intriguing

Lemma 1.1. *Suppose that $K \subset B \subset S$ are convex subsets of a Banach space X with K compact, S closed and bounded, and B open in S. If $T : S \to X$ is continuous, $\{T^j B, j \geq 0\} \subset S$ and K attracts points of B, then there is a closed, bounded, convex subset A of S such that*

$$A = \overline{\mathrm{co}}[\cup_{j\geq 1} T^j (B \cap A)] \qquad A \cap K \neq \emptyset.$$

Using Lemma 1.1 and Theorem 1.6, it is possible to prove

Lemma 1.2. *Suppose that $K \subset B \subset S$ are convex subsets of a Banach space X with K compact, S closed and bounded, and B open in S. If $T : S \to X$ is continuous, $\{T^j B, j \geq 0\} \subset S$, K attracts compact sets of B, and the set A of Lemma 1.1 is compact, then there is a fixed point of T.*

It is now easy to see that T being α-condensing implies that the set A of Lemma 1.1 is compact. To verify that the conditions of Lemma 1.2 are satisfied if T is compact dissipative requires some effort, and the verification uses arguments that are more closely related to stability theory (see Hale and Verduyn-Lunel (1993)).

As a simple application of Theorem 1.5, let us consider the RFDE

$$\dot{x}(t) = f(t, x_t) \tag{1.2}$$

where $x_t(\theta) = x(t+\theta), \theta \in [-r, 0]$, $r > 0$, and the initial data is chosen to be in the space $C = C([-r, 0] : \mathbb{R}^n)$. We assume that the function $f : \mathbb{R} \times C \to \mathbb{R}^n$ is smooth and that $f(t+\omega, \varphi) = f(t, \varphi)$ for all t, φ. The Poincaré map $T : C \to C$ is defined as $[T\varphi](\theta) = x(t+\theta, \varphi)$, $\theta \in [-r, 0]$, where $x(t, \varphi)$ is the solution of (1.2) with initial data φ on $[-r, 0]$. Let us assume that, for each fixed $t > 0$, the set $\{x(s, \varphi), 0 \leq s \leq t, \varphi \in B\}$ is bounded if B is bounded. Since a solution of (1.2) is continuously differentiable for $t \geq 0$, if $\omega \geq r$, then the Ascoli-Arzela theorem implies that T is completely continuous. If $\omega < r$, this is no longer true. On the other hand, if we write the solution $x(\cdot, \varphi)$ of (1.2) as

$$x_t(\cdot, \varphi) = S(t)\varphi + U(t)\varphi,$$

where $y_t \equiv S(t)\varphi$ satisfies

$$y(t) = 0 \text{ for } t \geq 0,$$
$$y(t) = \varphi(t), \text{ for } t \in [-r, 0],$$

then it is easy to see that $U(t)$ is a completely continuous operator. Furthermore, for any $\gamma > 0$, there is a $\beta > 0$ such that

$$\|S(t)\|_{\mathcal{L}(C;C)} \leq \beta e^{-\gamma t}, \quad t \geq 0.$$

Therefore, if we change the norm in C, the Poincaré map T is an α-contraction. Also, it is not too difficult to show that, if T is point dissipative, then it is compact dissipative. We can now use Theorem 1.5 to prove that there is a fixed point of T without any restriction on the period ω.

For other applications of Theorem 1.5 to NFDE, hyperbolic PDE, the beam equation, etc. see, for example, Hale (1988).

We end this section with some remarks about autonomous systems; that is, evolutionary equations on a Banach space X which generate a C^0-semigroup $T(t), t \geq 0$. In this case, for any $\tau > 0$, the map $T_\tau = T(\tau)$ is a Poincaré map. If we let $\mathrm{Fix}\,(T_\tau)$ be the set of fixed points of T_τ and we know that $\cup_{\tau \in (0,1]} \mathrm{Fix}\,(T_\tau)$ belong to a compact set, then it is not difficult to show that there is an equilibrium point for the semigroup; that is, a point $x_0 \in X$ such that $T(t)x_0 = x_0$ for all t.

To state a precise result, we use the concept of a global attractor. A set $\mathcal{A}$ in X is the *global attractor* for $T(t)$ if $\mathcal{A}$ is compact, invariant and attracts bounded sets of X. The *positive orbit* $\gamma^+(B)$ *of a bounded set* B *in* X is defined as the set $\{T(t)x : t \geq 0, x \in B\}$. The following result is contained in Hale (1988).

Theorem 1.7. *Suppose that $T(t) = S(t) + U(t) : X \to X$ is a C^0-semigroup, $U(t)$ is completely continuous and there is a continuous function $k : \mathbb{R}^+ \to \mathbb{R}^+$ such that $k(t,r) \to 0$ as $t \to \infty$ and $\|S(t)x\| \le k(t,r)$ if $\|x\| \le r$. If $T(t)$ is point dissipative and $\gamma^+(B)$ is bounded if B is bounded, then the global attractor $\mathcal{A}$ exists and there is an equilibrium point of $T(t)$.*

We only remark about the equilibrium point. The conditions imply that we can change the norm in X so that T_τ is an α-contraction and then $\mathrm{Fix}(T_\tau) \subset \mathcal{A}$ and we can use the remark preceding the theorem.

For autonomous ordinary differential equations, the existence of a zero of vector field can be proved under weaker conditions as in (i) and (ii) of Theorem 1.1.

An interesting example due to Jones and Yorke (1969) for autonomous ordinary differential equations in $\mathbb{R}^3$ shows that there is a vector field for which all solutions are bounded and yet there is no zero of the vector field. The construction makes use of the fact that there are minimal sets in $\mathbb{R}^3$ which do not separate $\mathbb{R}^3$ into two open sets with one bounded and the other unbounded (in particular, a torus can be a minimal set).

2. Ejective Fixed Point Theorems. Let us first define what we mean by an ejective point of a map.

Definition 2.1. Suppose that X is a Banach space, U is a subset of X and x is a given point in U. Given a map $T : U \setminus \{x\} \to X$, the point $x \in U$ is said to be an *ejective point of T* if there is an open neighborhood $G \subset X$ of x such that, for every $y \in G \cap U$, $y \ne x$, there is an integer $m = m(y)$ such that $T^m y \notin G \cap U$.

To motivate a general ejective fixed point theorem, we consider first some simple examples. Consider the ODE

$$\dot{x} = f(x), \tag{2.1}$$

where $x \in \mathbb{R}^2$ and f is a smooth function with $f(0) = 0$. Suppose that it is known that each solution oscillates about the origin; that is, if $x = (x_1, x_2)$ and $x(t)$ is a solution of (2.1) with initial data x_0 on the positive x_1-axis $K = \{(x_1, 0) : x_1 \in [0,\infty)\}$, then there is a first time τ for which $x(\tau) \in K$. The set K is a cone and, if we define $Tx_0 = x(\tau)$, then $T : K \to K$. The map T has a fixed point on the boundary ∂K of K given by $x = 0$, which corresponds to the equilibrium point 0 of (2.1). A nontrivial fixed point of T corresponds to a periodic orbit of (2.1). Without some additional conditions on the map T, zero may be the only fixed point. If we suppose that the origin is unstable, then there exists an $r > 0$ such that $|Tx| > |x|$ for $0 < |x| \le r$; that is, 0 is an ejective fixed point of T. Therefore, if, for $0 < |x| < r$, we know that the set $\{T^k x, k = 0, 1, 2, \ldots\}$ is bounded, then we will have a nontrivial fixed point of T. This will be true, for example, if we suppose that $|Tx| < |x|$ if $|x| \geq R$ for R sufficiently large. This is a simple ejective fixed point theorem.

These ideas are easily extended in principle to (2.1) where $x \in \mathbb{R}^3$. In fact, suppose that the equilibrium point 0 is hyperbolic with one real negative eigenvalue and two complex eigenvalues with positive real parts. Then the linearization of (2.1) about zero has a one dimensional stable manifold $W^s(0)$ and a two dimensional unstable manifold $W^u(0)$. Let us choose the coordinates $x = (x_1, x_2, x_3)$ so that $W^s(0) = \{(0,0,x_3), x_3 \in \mathbb{R}\}$ and $W^u(0) = \{(x_1, x_2, 0), (x_1, x_2) \in \mathbb{R}^2\}$. From the fact that the origin is a saddle point, there is a cone $\tilde{K}$ with nonempty interior (and therefore of three dimensions) which contains $W^u(0) \setminus \{0\}$ and is positively invariant under the flow for the linearization near 0. Also, becaues the eigenvalues with positive real parts are complex, the set $K \equiv \{x \in \tilde{K} : x = (x_1, 0, 0), x_1 \geq 0\}$ has the property that, for any $x_0 \in K$, there is a first return point $T_0 x_0$ to K under the linear flow. The set K is a cone with 0 being an ejective fixed point of T_0.

If we now suppose that the cone K also is positively invariant under the flow defined by (2.1) and, for any $x_0 \in K$, there is a first return time Tx_0 to K under the flow defined by (2.1), then we have $T : K \to K$ and 0 is an ejective fixed point of T. If we suppose also that there is a $R > 0$ such that $|Tx| < |x|$ for $|x| > R$, then we can define $K_R = \{x \in K : |x| \leq R\}$ and have $T : K_R \to K_R$ with 0 being an ejective fixed point. Using a small amount of theory of Liapunov functions, it is possible to deduce from the instability of zero that there is a $r > 0$ such that the set $K_{Rr} = \{x \in K_R : |x| \geq r\}$ has the property that $T : K_{Rr} \to K_{Rr}$. Brouwer's fixed point theorem imlies that there is a fixed point of T in K_{Rr} and, thus, a nontrivial periodic solution of (2.1).

The method described in the previous paragraphs actually has been applied to a specific problem in circuit theory concerned with the quenching of undesirable oscillations (see Oldenburger and Boyer (1961), Hale (1963)).

It is natural to attempt to carry the above procedure over to infinite dimensional systems by replacing the Brouwer theorem by the Schauder theorem for compact maps. Unfortunately, it is not so easy to exclude a small neighborhood of the origin using Liapunov theory. Convexity (or being equivalent to a convex set) is not easy to verify. Motivated by an example to be discussed below, Browder (1965) gave an ejective fixed point theorem for compact maps, which was later extended by Nussbaum (1974) to the following result.

Theorem 2.1. *If K is a closed, bounded, convex, infinite dimensional set in X, $T : K \setminus \{x_0\} \to K$ is an α-contraction, and x_0 is an ejective point of T, then there is a fixed point of T in $K \setminus \{x_0\}$. If K is finite dimensional and x_0 is an extreme point of K, then the same conslusion holds.*

By combining in a very interesting way the concept of eigenvalue for cone maps as in the work of Krasnoselskii (1964) and Grafton (1969), Nussbaum (1974) proved the following more global result. For any $M > 0$, let $S_M = \{x \in X : |x| = M\}$, $B_M = \{x \in X : |x| \leq M\}$.

Theorem 2.2. *If K is a closed convex set in X, $T : K \setminus \{0\} \to K$ is an α-contraction, $0 \in K$ is an ejective point of K, and there is an $M > 0$ such that $Tx = \lambda x$, $x \in K \cap S_M$, implies $\lambda < 1$, then T has a fixed point in $K \cap B_M \setminus \{0\}$ if either K is infinite dimensional or X is finite dimensional and 0 is an extreme point of K*

In specific application of these results to functional differential equations, it is possible to take advantage of some of the geometric properties of the flow in order to obtain the ejectivity of the special point in the statement of the above theorems. We state this precisely in the proper context.

Let us now turn to a specific delay differential equation, referred to as Wright's equation (Wright (1955), (1961)), which was studied in detail by Jones (1962) and was the motivation for the development of the above general ejective fixed point theorems.

Suppose that α is a positive constant and consider the equation

$$\dot{x}(t) = -\alpha x(t-1)[1 + x(t)]. \tag{2.2}$$

If $C = C([-1,0], \mathbb{R})$, then, for any $\varphi \in C$, there is a unique solution $x(t, \varphi, \alpha)$ of (2.2) satisfying $x(\theta, \varphi, \alpha) = \varphi(\theta), \theta \in [-1, 0]$. If we assume that $\varphi(0) > -1$, then it is easy to see that $x(t, \varphi, \alpha)$ exists for all $t \geq 0$ and $x(t, \varphi, \alpha) > -1$ for all $t \geq 0$. In the following, we will always assume that $\varphi(0) > -1$ and still use the notation $\varphi \in C$. If we define $[T_\alpha(t)\varphi](\theta) = x(t + \theta, \varphi, \alpha)$, $\theta \in [-1, 0]$, then $T_\alpha(t), t \geq 0$, is a C^0-semigroup on C.

Our objective is to determing periodic orbits of (2.2). Let us try to mimic the procedure outlined above for (2.1) for $n = 3$. To do this, we first must understand the behavior of the solutions of the linear variational equation about the origin:

$$\dot{x}(t) = -\alpha x(t-1). \tag{2.3}$$

The eigenvalues of this equation are those values of λ for which there is a solution $e^{\lambda t}$; that is, those λ which satisfy the *characteristic equation*

$$\lambda + \alpha e^{-\lambda} = 0. \tag{2.4}$$

This equation has infinitely many solutions with $\operatorname{Re} \lambda \to \infty$ as $|\lambda| \to \infty$. Other properties of these eigenvalues is given in the following result.

Lemma 2.1.

(i) If $0 < \alpha < \pi/2$, every solution of (2.4) has negative real part.

(ii) If $\alpha > e^{-1}$, there is a solution $\lambda(\alpha) = \gamma(\alpha) + i\sigma(\alpha)$ of (2.4) which is continuous together with its first derivative in α, $\gamma'(\alpha) > 0$, $\sigma'(\alpha) > 0$, $0 < \sigma(\alpha) < \pi$, $\sigma(\pi/2) = \pi/2$, $\gamma(\pi/2) = 0$, and $\gamma(\alpha) > 0$ for $\alpha > \pi/2$.

Lemma 2.1 implies that there is a Hopf bifurcation at $\alpha = \pi/2$. It is possible to show that it is supercritical (that is, in a neighborhood of $x = 0$, $\alpha = \pi/2$, there is a periodic orbit only if $\alpha > \pi/2$, it is unique, hyperbolic and stable) (see Chow and Mallet-Paret (1977)). Since $\sigma(\pi/2) = \pi/2$, the bifurcating periodic solution has period approximately 4 and is ***slowly oscillating***; that is, the distance between zeros of the solution is greater than the delay (> 1).

As α increases, there are more eigenvalues which cross the imaginary axis and lead to Hopf bifurcations. However, these periodic solutions are not slowly oscillating. This suggests that, if we attempt to carry out a program similar to the mentioned one for ODE, we must restrict attention to the set SO of slowly oscillation solutions of (2.2).

For all values of α, the dominant unstable eigenvalues are the ones in part (ii) of Lemma 2.1. The corresponding eigenfunctions span a two dimensional subspace W of C and there is a cone $\tilde{K}$ in C with nonempty interior, $W \setminus \{0\}$ belongs to the interior of $\tilde{K}$ and $\tilde{K}$ is positively invariant under the flow defined by the linear equation (2.3). Since the dominant eigenvalues are complex, solutions of (2.3) rotate around K. If we knew that the same properties hold for the nonlinear equation (2.2), then we would be close to situation in ODE. All of these remarks can be made more precise and we do this following the ideas in Jones (1962). However, we first state the main result for (2.2).

Theorem 2.3. *If $\alpha > \pi/2$, then (2.2) has a nontrivial periodic solution which is slowly oscillating.*

We only give the main steps in the proof and refer to Hale and Verduyn-Lunel (1993) for the details. We first need a result on the oscillatory properties of the solutions of (2.2).

Lemma 2.2.

(i) If $\varphi(0) > -1$ and the zeros of $x(\cdot, \varphi, \alpha)$ are bounded, then $x(t, \varphi, \alpha) \to 0$ as $t \to \infty$.

(ii) If $\varphi(0) > -1$, then $x(t, \varphi, \alpha)$ is bounded. Furthermore, if the zeros of $x(\cdot, \varphi, \alpha)$ are unbounded, then any maximum of $x(t, \varphi, \alpha)$, $t > 0$ is less that $e^{\alpha} - 1$.

(iii) If $\varphi(0) > -1$ and $\alpha > 1$, then the zeros of $x(\cdot, \varphi, \alpha)$ are unbounded.

(iv) If $\varphi(\theta) > -1$, $-1 < \theta < 0$, then the zeros of $x(t, \varphi, \alpha)$ are simple and the distance from a zero of $x(t, \varphi, \alpha)$ to the next maximum or minimum is ≥ 1.

Let K be the class of all monotone increasing functions $\varphi \in C$ such that $\varphi(\theta) > 0$, $-1 < \theta \leq 0$, $\varphi(-1) = 0$. Also, suppose that $0 \in K$. The set K is a cone. If $\alpha > 1$, $\varphi \in K$, $\varphi \neq 0$, let

$$z(\varphi, \alpha) = \min\{t : x(t, \varphi, \alpha) = 0, \dot{x}(t, \varphi, \alpha) > 0\}.$$

This minimum exists from Lemma 2.2, Parts (iii) and (iv). Also $z(\varphi,\alpha) > 2$. Furthermore, Lemma 2.2(iv) implies that $x(t,\varphi,\alpha)$ is positive and increasing on $(z(\varphi,\alpha), z(\varphi,\alpha)+1]$. As a consequence, if $\tau(\varphi,\alpha) = z(\varphi,\alpha)+1$, then we can define the mapping $A : K \to K$ by

$$\begin{aligned} &A(\alpha)0 = 0 \\ &A(\alpha)\varphi = T_\alpha(\tau(\varphi))(\varphi), \quad \varphi \neq 0, \end{aligned}$$

where $T_\alpha(t)$ is the semigroup defined by (2.2).

Lemma 2.3. *The map $\tau : K \setminus \{0\} \times (1,\infty) \to (0,\infty)$ defined by $\tau(\varphi,\alpha) = z(\varphi,\alpha)+1$ is completely continuous.*

From Parts (ii) and (iv) of Lemma 2.2, it follows that $|A(\alpha)\varphi| \leq e^\alpha - 1$ for each $\varphi \in K$ and $A(\alpha)$ takes any bounded set B in $K \setminus \{0\}$ into the set $\{\varphi \in C : |\varphi| \leq e^\alpha - 1\}$. Using Lemma 4.3 and the fact that $A(0) = 0$, it is possible to show that $A(\alpha)$ is continuous at 0 and that $A(\alpha)$ is completely continuous.

If we let $K_\alpha = \{\varphi \in K : |\varphi| \leq e^\alpha - 1\}$, then $A : K_\alpha \to K_\alpha$ is completely continuous with the fixed point 0 an extreme point of K_α. Also, the set K_α is a closed bounded convex subset of C. If the point 0 is ejective, then we can use Theorem 2.1 to complete the proof of Theorem 2.3. The proof of this is a little technical, but follows from the fact that the dominant part of the local unstable manifold of the origin is tangent at the origin to the two dimensional manifold W of the dominant part of the unstable manifold of the linear equation (2.3). We do not give the details.

If, in (2.2), we let $1 + x(t) = e^{y(t)}$, then we obtain the equation

$$\dot{x}(t) = \alpha f(y(t-1)), \tag{2.5}$$

where $f(y) = 1 - e^y$ which satisfies the property that $f(y)$ has negative feedback; that is, $yf(y) < 0$ for $y \neq 0$ and $f'(0) < 0$. For equations of the form (2.5) with some additional conditions on the function f, the above method has been applied to obtain the existence of nontrivial periodic solutions (see Hale and Verduyn-Lunel (1993), Supplementary Remarks to Chapter 11).

Equations (2.5) and more general ones of the type

$$\dot{x}(t) + \sigma x(t) = f(x(t-1)) \tag{2.6}$$

occur very often in the applications. The following interesting result is due to Hadeler and Tomiuk (1977).

Theorem 2.4. *Suppose that σ is a positive constant and f has negative feedback. If there is an interval I such that $f(I) \subset I$ and the origin is linearly unstable, then there is a nontrivial periodic solution of (2.6) which is slowly oscillating.*

The above method will not yield directly a proof of this theorem since the slowly oscillating periodic solution need not be monotone increasing of an interval of length one of the form $[z, z+1]$, where z is a zero of the solution. The important new idea is to replace the cone K used for the proof of Theorem 2.3 by another consisting of functions which when weighted by a specific type of exponential function are monotone increasing.

Let us condider the following generalization of (2.6) which has been used as a model for the transmission of light through a ring cavity (see Vallee, Dubois, Coté and Delisle (1987), Vallee and Marriott (1989))

$$(2.7) \qquad (\delta_m \frac{d}{dt} + 1) \cdots (\delta_1 \frac{d}{dt} + 1) y(t) = f(y(t-1)),$$

where $\delta = (\delta_1, \ldots, \delta_m) \in (0, \infty)^m$. Hale and Ivanov (1993) have used the ideas of Hadeler and Tomiuk (1977) as well as the method above to obtain the following result.

Theorem 2.5. *Suppose that I is a bounded interval such that $f(I) \subset I$, f has negative feedback. Then there is a $\delta_0 > 0$ such that, for each $\delta \in (0, \delta_0)^m$, equation (2.7) has a slowly oscillating periodic solution.*

Theorem 2.5 is Theorem 2.4 if $m = 0$. For $m = 2$, an der Heiden (1979) has obtained the conclusion in Theorem 2.5 with any smallness restrictions on ϵ but, of course, with the additional condition that the origin is linearly unstable.

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Part 2. Large Delays and Oscillations

1. Introduction. It is a well known fact that increasing the delay in a retarded delay differential equation (RDDE) often has a tendency to destabilize the motion. Also, the destabilization frequently occurs through a Hopf bifurcation; that is, as the delay is increased, a periodic orbit bifurcates from an equilibrium point. We have encountered this phenomenon in our study of Wright's equation (2.2) of Part1, Section 2. By a rescaling in time, the parameter α in (2.2) plays the same role as a delay. We now want to discuss situations where the destabilization occurs through a Hopf bifurcation and try to understand the nature of the profile of the resulting periodic orbits as the delay approaches infinity.

We begin with a simple example of a linear system to see how the eigenvalues behave as a function of the delay.

Example 1.1. For a, b constants, consider the RDDE

$$\dot{x}(t) = -ax(t) - bx(t-1). \tag{1.1}$$

As in Part 1, the initial data for solutions is taken in the space $C = C([-1, 0]; \mathbb{R})$. Also, if $x(t, \varphi)$ is the solution of (1.1) with initial data $\varphi \in C$ at $t = 0$, then we define the C^0-semigroup $T(t) : C \to C$ by the relation $T(t)\varphi(\theta) = x(t + \theta, \varphi), \theta \in [-1, 0]$. The operator $T(t)$ is compact for $t \geq 1$, the spectrum of $T(t)$ consists only of point spectrum and it is given by $\{e^{\lambda t}\}$, where λ is a solution of the characteristic equation

$$\lambda + a + be^{-\lambda} = 0. \tag{1.2}$$

In the (a, b)-plane, the stability diagram is show in Figure 1.1.

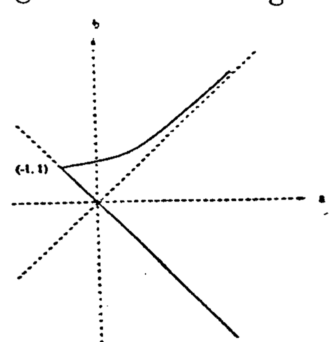

Figure 1.1. Stability region for $\dot{x}(t) = -ax(t) - bx(t-1)$.

In the region between the line $b = -a$ and the curve asymptotic to the line $a = b$ which contains the wedge $|a| < |b|, a > 0$, each solution of (1.2) has negative real part and the zero solution of (1.1) is exponentially stable. On the line $b = -a$, $(a, b) \neq (-1, 1)$, there is a simple zero of (1.2). On the curve asymptotic to the line $a = b$, $(a, b) \neq (-1, 1)$, there are two purely imaginary roots $\pm i\omega(a, b)$ with $\omega(a, b) \to \pi$ as $(a, b) \to (\infty, \infty)$. At the point $(a, b) = (-1, 1)$, there is a double zero root with nonsimple elementary divisors (see, for example, the Appendix of Hale and Verduyn-Lunel (1993)).

For $\tau > 0, \bar{a}, \bar{b}$ constants, consider the equation

$$\dot{x}(t) = -\bar{a}x(t) - \bar{b}x(t-\tau), \tag{1.3}$$

which by a change of time scale is equivalent to the equation (1.1) with $a = \bar{a}\tau, b = \bar{b}\tau$. If we suppose that $(\bar{a}, \bar{b})$ is in the stability region for (1.1), then we can destabilize the origin by increasing the delay if and only if $\bar{b} > |\bar{a}|$. The destabilization occurs by having two eigenvalues cross transversally the imaginary axis at a critical value $\tau(\bar{a}, \bar{b}) > 0$. If we subject (1.1) to nonlinear perturbations which are second order near the origin, then the Hopf bifurcation theorem implies there must be periodic orbits in a neighborhood of $(x, \tau) = (0, \tau(\bar{a}, \bar{b})$.

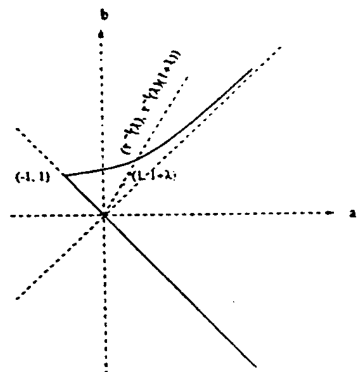

Figure 1.2. Stability of the origin of (1.3) for $(\bar{a}, \bar{b}) = (1, 1+\lambda))$.

To be more specific, let us assume that $\bar{a} = 1$, $\bar{b} = 1 + \lambda$, where λ is a real parameter. In this case, we can destabilize the origin by increasing τ if $\lambda > 0$ and cannot if $\lambda \leq 0$. Also, if $\lambda > 0$ and if we define $\tau(\lambda) \equiv \tau(1, 1+\lambda)$, then $\tau(\lambda) \to \infty$ as $\lambda \to 0$ and the purely imaginary solutions $\pm i\omega(\lambda, \tau(\lambda))$ of (1.2) are such that $\omega(\lambda, \tau(\lambda)) \to \pi$ as $\lambda \to 0$. Note that the corresponding eigenfunction is periodic of period $\frac{2\pi}{\omega(\lambda, \tau(\lambda))}$ which approaches 2 as $\lambda \to 0$.

If we keep $\bar{a} = 1$, $\bar{b} = 1 + \lambda$ and formally put $\tau = \infty$ in the rescaled version of (1.3), we obtain the mapping on $\mathbb{R}$: $x \mapsto -(1+\lambda)x$. The fixed point zero of this map is stable for $\lambda < 0$ and unstable for $\lambda > 0$. Furthermore, if nonlinear terms were included, then there will be a period doubling at $\lambda = 0$.

Let us now consider the nonlinear equation

$$\epsilon\dot{x}(t) + x(t) = f_\lambda(x(t-1)), \tag{1.4}$$

where

$$f_\lambda(x) = -(1+\lambda)x + g(x), \quad g(x) = O(x^2) \text{ as } x \to 0. \tag{1.5}$$

Recall that $\epsilon = 1/\tau$, where τ is the delay.

Let $(\lambda, \epsilon(\lambda))$ be the curve in the (λ, ϵ)-plane along which there are two purely imaginary solutions of (1.2) and the remaining ones have negative real parts. For a fixed small value of λ, the point $\epsilon(\lambda)$ is a point of Hopf bifurcation with respect to ϵ for (1.4). In the (λ, ϵ)-plane, the Hopf bifurcation curve divides the upper half plane intersected with a neighborhood of the origin into two regions S and U with the property that the origin is hyperbolic stable in S and hyperbolic unstable in U (see Figure 1.3) provided that we restrict our discussion to initial data from the subspace SO of C which corresponds to slowly oscillating solutions. Recall that a slowly oscillating solution is one for which the distance between zeros is at least as large as the delay, which in our equation is 1. In the subspace SO, the dimension of the unstable manifold of the origin in U is 2.

We restrict our attention to SO because, for a fixed value of λ, there are a countable number of values of ϵ at which two eigenvalues cross the imaginary axis from left to right with increasing ϵ.

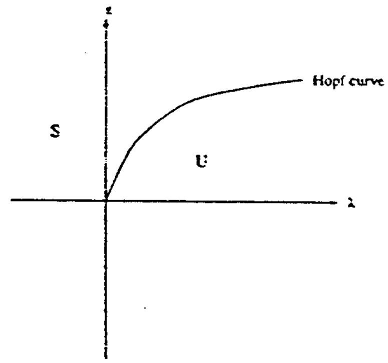

Figure 1.3. The stable and unstable regions of the origin.

Let $x^{\epsilon\lambda}$ be a slowly oscillating periodic orbit of (1.4) in a small neighborhood of $x = 0$ for ϵ, λ small, $\epsilon > 0$. The problem that we want to discuss is the following:

For a fixed value of λ, what is the limiting profile of this solution as $\epsilon \to 0$?

From our remarks above, we know that the period is approximately 2 since the eigenvalues $\pm i\omega(\lambda, \tau(\lambda))$ on the Hopf curve have the property that $\omega(\lambda, \tau(\lambda)) \to \pi$ as $\lambda \to 0$.

For $\epsilon = 0$ in (1.4), we obtain the map $x \to f_\lambda(x)$. It is natural to conjecture that there should be some relationship between the dynamics of (1.4) near the origin and the map f_λ near the origin. In particular, we might expect the periodic orbits to be related to period two points of the map.

The point $\lambda = 0$ is a point of bifurcation to period two points of the map f_λ. Let us suppose that, in a small neighborhood of $x = 0$, $\lambda = 0$, there are only a finite number of period two points $(\alpha_j^\lambda, \beta_j^\lambda)$, $\alpha_j^\lambda > 0$, $j = 1, 2, \dots, N$, all hyperbolic, and ordered as $\alpha_j^\lambda < \alpha_k^\lambda$ if $j < k$. Define $(\alpha_0^\lambda, \beta_0^\lambda) = (0, 0)$. The period two points of f_λ will alternate their stability properties; that is, if $(\alpha_j^\lambda, \beta_j^\lambda)$ is stable, then $(\alpha_{j-1}^\lambda, \beta_{j-1}^\lambda)$ and $(\alpha_{j+1}^\lambda, \beta_{j+1}^\lambda)$ are unstable.

To each $(\alpha_j^\lambda, \beta_j^\lambda)$, we can define a *square wave* 2-periodic function $s_j^\lambda(t)$ by the relation

$s_j^\lambda(t) = \alpha_j^\lambda$ (resp. β_j^λ) if $0 \le t < 1$ (resp. $1 \le t < 2$) (see Figure 1.4a). This function is stable (resp. unstable) for the map f_λ if the period two point $(\alpha_j^\lambda, \beta_j^\lambda)$ is stable (resp. unstable) for the map f_λ.

When the period two point $(\alpha_j^\lambda, \beta_j^\lambda)$, $j \ge 1$, is unstable, the square wave 2-periodic function s_j^λ has an infinite dimensional unstable manifold. Therefore, we do not expect it to be related to any periodic orbit that begins near a Hopf curve. Because of the nature of the solutions of the characteristic equation (1.2), there is a two dimensional center manifold of (1.4) for each (λ, ϵ) near $(\lambda_0, \epsilon(\lambda_0))$. If (λ, ϵ) is small, we also expect that there should be some two dimensional invariant manifold that contains all slowly oscillating periodic orbits near the origin with each periodic orbit on this manifold encircling the origin. If each such periodic orbit is hyperbolic, then they alternate between being stable and unstable; that is, the index of the corresponding Poincaré map is either zero or 1. It is to be expected under reasonable conditions that the limiting periodic orbits as $\epsilon \to 0$ should be related to the period two points of the map f_λ.

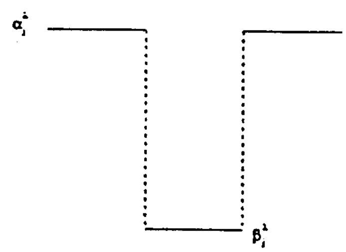

Figure 1.4a. Square wave s_j^λ

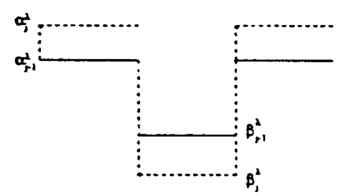

Figure 1.4b. Pulse wave p_j^λ

As a consequence of the above intuitive remarks, when the period two point $(\alpha_j^\lambda, \beta_j^\lambda)$, $j \ge 1$, is unstable, we define a *pulse* 2-periodic function $p_j^\lambda(t)$ by the relation $p_j^\lambda(t) = \alpha_j^\lambda$ (resp. β_j^λ) if $t = 0$ (resp. $t = 1$), $p_j^\lambda(t) = \alpha_{j-1}^\lambda$ (resp. β_{j-1}^λ) if $0 < t < 1$ (resp. $1 < t < 2$) (see Figure 1.4b). The pulse 2-periodic function is unstable for the map f_λ and the dimension of the unstable manifold is 1 (if we exclude translations in t by a constant).

Following our intuition that, for ϵ small, we expect that the orbits of (1.4) near the origin should behave in some way as the orbits of the map f_λ, we make the following conjecture.

Conjecture. *For ϵ, λ small and in a sufficiently small neighborhood of the origin, for each $j > 1$, there exists a periodic solution $x_j^{\epsilon\lambda}$ of (1.4) of period approximately 2 such that*

(1) $x_j^{\epsilon\lambda} - s_j^\lambda \to 0$ as $\epsilon \to 0$ uniformly on compact subsets of $\mathbb{R} \setminus Z$ if $(\alpha_j^\lambda, \beta_j^\lambda)$ is stable,

(2) $x_j^{\epsilon\lambda} - p_j^\lambda \to 0$ as $\epsilon \to 0$ uniformly on compact subsets of $\mathbb{R} \setminus Z$ if $(\alpha_j^\lambda, \beta_j^\lambda)$ is unstable, where $Z = \{n : n = 0, \pm 1, \pm 2, \ldots\}$.

If the intuitive remarks made before the statement of the conjecture could be made precise, one could probably prove the conjecture. Unfortunately, we do not know at this time how to do this.

Another possible approach to a proof could be the following. At each point $(\lambda_0, \epsilon(\lambda_0))$ on the Hopf bifurcation curve, it is possible to use the method of Liapunov-Schmidt to obtain the bifurcation function $G(\lambda, \epsilon, c)$ for (λ, ϵ) in a neighborhood of $(\lambda_0, \epsilon(\lambda_0))$ and for the approximate amplitude c of the periodic solutions of period approximately $2\pi/\omega(\lambda_0, \epsilon(\lambda_0))$, where $i\omega(\lambda_0, \epsilon(\lambda_0))$ is the purely imaginary eigenvalue on the Hopf bifurcation curve. We know that $\omega(\lambda_0, \epsilon(\lambda_0)) \to \pi$ as $\lambda_0 \to 0$. The zeros of $G(\lambda, \epsilon, c)$ correspond to periodic solutions of (1.4) with period close to 2. Therefore, it is to be expected that these zeros should be related to the periodic points of period 2 of the map f_λ; that is, to the fixed points of the map f_λ^2. If this could be proved, we would be very close to a proof of the conjecture.

For the case in which there is a generic period doubling for the map f_λ, the above conjecture is known to be true. Let us state the result more precisely. Suppose that

$$f_\lambda(x) = -(1+\lambda)x + ax^2 + bx^3, \quad \beta = a^2 + b \neq 0. \tag{1.6}$$

We remark that it is not necessary to assume that f_λ is a polynomial in x or that the dependence upon λ is as simple as in (1.6). In fact, it is enough to assume that f_λ has the property that

$$f_\lambda(0) = 0 \text{ for all } \lambda,$$
$$\partial_x f_0(0) = -1, \quad \partial^2_{x\lambda} f_0(0) = -1,$$
$$\partial^2_x f_0(0) = a, \quad \partial^3_x f_0(0) = b.$$

We assume (1.6) only for some simplicity in notation.

Under the assumption (1.6) on f_λ, the map f_λ undergoes a generic period doulbling at $\lambda = 0$. In fact, there is a neighborhood $V \times W \subset \mathbb{R}^2 \times \mathbb{R}$ of $(0,0) \in \mathbb{R} \times \mathbb{R}$ such that, for each value of $\lambda \in W$ for which $\lambda\beta > 0$, there is a unique period two point $(\alpha_\lambda, \beta_\lambda)$ of f_λ with $\alpha_\lambda, \beta_\lambda \in V$. We say that the bifurcation is *supercritical* if $\beta > 0$ (the fixed point 0 is stable for f_0) and *subcritical* if $\beta < 0$ (the fixed point 0 is unstable for f_0) (see Figure 1.5). This implies that the period two point is hyperbolic stable (resp. unstable) if $\beta > 0$ (resp. $\beta < 0$).

Figure 1.5. Bifurcation diagram for f_λ

Theorem 1.1. *Suppose that S, U are the stable and unstable regions of the origin associated with the Hopf bifurcation curve and that f_λ satisfies (1.6). Then there is a neighborhood V of $(0, 0)$ in the (λ, ϵ)-plane and a neighborhood W of $x = 0$ such that, if $\beta > 0$ (resp. $\beta < 0$) and $(\lambda, \epsilon) \in V$, then there is a periodic solution $x^{\lambda,\epsilon}$ of (1.4) in W with period $2\tau(\lambda, \epsilon) = 2 + 2\epsilon + O(|\epsilon|(|\lambda| + |\epsilon|))$ as $(\lambda, \epsilon) \to (0, 0)$ if and only if $(\lambda, \epsilon) \in V \cap U$ (resp. $(\lambda, \epsilon) \in V \cap S$). Furthermore, this solution is unique and*

(1) $x^{\lambda,\epsilon} - s^\lambda \to 0$ as $\epsilon \to 0$ uniformly on compact subsets of $\mathbb{R} \setminus \mathbb{Z}$ if $\beta > 0$,

(2) $x^{\lambda,\epsilon}$ is pulse-like and $x^{\lambda,\epsilon} - p^\lambda \to 0$ as $\epsilon \to 0$ uniformly on compact subsets of $\mathbb{R} \setminus \mathbb{Z}$ if $\beta < 0$,

where s^λ, p^λ are respectively the square and pulse 2-periodic functions defined above. If, in addition, $f_\lambda(z) = -f_\lambda(-z)$, then $x^{\lambda,\epsilon}(t + \tau(\lambda, \epsilon)) = -x^{\lambda,\epsilon}(t)$.

In the case $\beta < 0$, the meaning of pulse-like is that the solution $x^{\lambda,\epsilon}$ limits to constants on the integers. However, the values of these constants exceed the values of the corresponding period two points of the map. Theorem 1.1 is due to Chow, Hale and Huang (1992) in the supercritical case and to Hale and Huang (1992a) for the subcritical case.

It is possible to make further reasonable conjectures about the relationship between the map f_λ and equation (1.4) for ϵ, λ small. More specifically, we should consider the parameter λ as a vector and obtain a generic unfolding of a codimension q singularity for period 2 points of the map f_λ and then obtain a relationship between the bifurcation surfaces in λ-space for period two points and the bifurcation surfaces in (ϵ, λ)-space for periodic solutions of (1.4) of period approximately 2.

In the next section, we outline the proof of Theorem 1.1. In Section 3, we given some extensions to matrix equations and, in Section 4, we given further extensions to the case where we have differential equations coupled with difference equations.

There are results concerning the existence of periodic solutions which are not necessarily small. In $f_\lambda(x)$ represents a negative feedback, $xf_\lambda(x) < 0$ for $x \neq 0$ and leaves an interval invariant (Theorem 2.4, Part 1), there exist a nontrivial periodic solution for every $\epsilon > 0, \lambda > 0$. Under some additional conditions, the limiting profile as $\epsilon \to 0$ is a square wave (see Mallet-Paret and Nussbaum (1986a)). When related to Theorem 1.1, these conditions correspond to the case $\beta > 0$; that is, the orbit is stable. No global results which correspond to the case $\beta < 0$ are known.

The conjecture above relating the periodic solutions of (1.4) to the period two points of the map f_λ is known not to hold in general if the solutions are not required to remain in a small neighborhood of the origin. Furthermore, in this global setting, it is generally not possible to associate period k points of the map with solutions of (1.4) (see Mallet-Paret and Nussbaum (1986b), (1994)). On the other hand, these global results do not rule out the possibility of the validity of the conjecture.

2. Proof of Theorem 1.1. We now give an outline of the ideas of the proof of Theorem 1.1, omitting the nontrivial technical details although they have independent interest.

At a point (λ_0, ϵ_0) on the Hopf bifurcation curve, there are two purely imaginary solutions $\pm i\omega_0$ of (1.2) and the remaining solutions have negative real parts. It is possible to extend the classical transformation theory (theory of normal forms) in ODE for determining the approximate flow on the center manifold of (1.4) at $(\lambda, \epsilon) = (\lambda_0, \epsilon_0)$ corresponding to these purely imaginary eigenvalues. If we assume that the function f_λ is given by (1.6) with $\beta \neq 0$ and these computations are performed, then the stability properties of the origin under the mapping f_λ are determined by β and the origin is stable (resp. unstable) if $\beta > 0$ (resp. $\beta < 0$), which corresponds to the generic supercritical (resp. subcritical) bifurcation of the map f_λ at $\lambda = 0$. Once this normal form has been obtained, then the fact that the solutions of (1.3) close to $\pm i\omega_0$ cross the imaginary axis from right to left as ϵ decreases implies that there is a generic supercritical (resp. subcritical) Hopf bifurcation with respect to ϵ at (λ_0, ϵ_0). Therefore, in a neighborhood of any point (λ_0, ϵ_0) on the Hopf bifurcation curve, there is a unique periodic orbit in the region S (resp. U) if $\beta > 0$ (resp. $\beta < 0$) which is stable (resp. unstable). Of course, we do not know if this orbit exists and is unique in all of S (resp. U); of course, in a small neighborhood of the origin. It will be one of tasks to show that this is the case.

Normal form theory for retarded functional differential equations is an interesting and nontrivial subject in itself and we refer the reader to the papers of Faria and Magalhães (1991), (1992) for the theory as well as applications.

We remark that we could have obtained the same information about the generic Hopf bifurcation by using the method of Liapunov-Schmidt to obtain the bifurcation function for the periodic orbits in a neighborhood of the origin. For a discussion of this approach, see, for example, Stech (1979), (1985).

Returning to our proof, we next show the existence of the periodic orbit in S (resp. U) and then determine the limiting profile of the periodic solution as $\epsilon \to 0$. Since we expect to have some fast changes in the shape of the periodic solution near the integers, we enlarge the system by introducing new variables which represent separately the solution on the interval $[0, 1 + r\epsilon)$ and $[1 + r\epsilon, 2 + 2r\epsilon)$. Our tranformation also will change the bifurcation problem in the 'bad' parameters (λ, ϵ) (bad because we must consider $\epsilon > 0$ and cannot consider it in a neighborhood of zero) into a bifurcation problem with 'good' parameters (λ, r) (good in the sense that (λ, r) can be considered in a full neighborhood of some point). To accomplish this, we introduce some scalings which were originally proposed by S.-N. Chow several years ago. We suppose that (1.4) has a periodic solution

$z(t)$ with period $2+2r\epsilon$ and we let

$$w_1(t) = z(-\epsilon r t), \quad w_2(t) = z(-\epsilon r t + 1 + \epsilon r). \tag{2.1}$$

Since $z(t)$ has period $2+2r\epsilon$, we see that

$$\begin{aligned} w_2(t) &= z(-\epsilon r(t+1) - 1) \\ w_2(t-1) &= z(-\epsilon r t - 1). \end{aligned} \tag{2.2}$$

If we use (2.1) and (2.2) in (1.4), we deduce that

$$\begin{aligned} \dot w_1(t) &= r w_1(t) - r f_\lambda(w_2(t-1)) \\ \dot w_2(t) &= r w_2(t) - r f_\lambda(w_1(t-1)). \end{aligned} \tag{2.3}$$

This equation is now independent of ϵ. We remark that, if (w_1, w_2) is a solution of (2.3), then (w_2, w_1) also is a solution of (2.3).

We next determine the approximate value of the constant r in the period $2+2r\epsilon$. This is obtained by considering the linear variational equation around the zero solution of (2.3) for $\lambda = 0$,

$$\begin{aligned} \dot w_1(t) &= r w_1(t) + r w_2(t-1) \\ \dot w_2(t) &= r w_2(t) + r w_1(t-1). \end{aligned} \tag{2.4}$$

The eigenvalues of (2.4) are the roots of the characteristic equation,

$$\det(\mu I - r L e^{\mu\cdot} I) = (\mu - r)^2 - r^2 e^{-2\mu} = 0, \tag{2.5}$$

where

$$L\varphi = \varphi(0) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \varphi(-1). \tag{2.6}$$

The left hand side of Equation (2.5) always has $\mu = 0$ as a zero. It is a simple zero if $r \neq 1$ and it is a double zero if $r = 1$. Bifurcation from a simple zero can never lead to any periodic orbits. Therefore, we are forced to take $r = 1$ in the first approximation. For $r = 1$, the remaining eigenvalues of (2.4) have negative real parts. If we let $r = 1 + \delta$, $w = (w_1, w_2)$, where δ is a small parameter, then (2.3) can be written as

$$\dot w(t) = L w_t + \delta L w_t - F_{\lambda,\delta}(w_t), \tag{2.7}$$

where

$$F_{\lambda,\delta}(\varphi) = (1+\delta) \begin{bmatrix} \varphi_2(-1) + f_\lambda(\varphi_2(-1)) \\ \varphi_1(-1) + f_\lambda(\varphi_1(-1)) \end{bmatrix} \qquad \varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}, \tag{2.8}$$

and $w_t(\theta) = w(t+\theta)$ for $-1 \le \theta \le 0$.

We now consider Equation (2.7) as a perturbation of the linear equation

$$\dot{v}(t) = Lv_t\,. \tag{2.9}$$

Of course, we will consider (2.7) with initial data in the space $C \equiv C([-1, 0], \mathbb{R}^2)$. Since the characteristic equation for the linear part of (2.7) for $(\lambda, \delta) = (0, 0)$ has a zero as a root of multiplicity two, we know that the small periodic orbits of (2.7) will lie on a two dimensional center manifold which is tangent to the subspace generated by generalized eigenvectors ζ_1, ζ_2 associated with the eigenvalue zero of (2.8).

To obtain a center manifold, we use the usual decomposition theory for linear systems (see Hale and Verduyn-Lunel (1993)) writing $C = P \oplus Q$, where P is the generalized eigenspace of 0 and Q is a subspace complementary to P and invariant under the semigroup defined by (2.9). If we let $w_t = z_1\zeta_1 + z_2\zeta_2 + \bar{w}_t$, where $\bar{w}_t \in Q$, then a center manifold is given by a function $h(\zeta_1, \zeta_2)$ which vanishes at $(0,0)$ and is defined by an integral equation which is obtained from the variation of constants formula. To solve our problem, we need to know the specific form of the vector field on the center manifold. This is a nontrivial computation very similar in spirit to those that are involved in nomal form theory. If we perform these computations, we see that the approximate flow on the center manifold is given by the system of ordinary differential equations

$$\begin{aligned} \dot{z}_1 &= 2\delta z_1 + 2\lambda(\frac{2}{3}z_1 + z_2) - 2\beta(\frac{2}{3}z_1 + z_2)^3 - a^2(\frac{2}{3}z_1 + z_2)z_1^3 \\ \dot{z}_2 &= -z_1 \end{aligned} \tag{2.10}$$

up through terms of order $(\delta+\lambda)^2|z| + |z|^4$.

Because of the symmetry in (2.3), it can be shown that the periodic orbits of (2.10) which encircle the origin and have period > 2 are in one-to-one correspondence with the periodic solutions of (1.4) of period $2 + 2r\epsilon$. Of course, in this statement, we assume that ϵ is small as well as the periodic orbits that are being considered. With this observation, it is sufficient to look for these types of periodic solutions of (2.10) in a neighborhood of the origin regarding (λ, δ) as the bifurcation parameters. We remark that this problem is different from the usual bifurcation problem for periodic orbits since we are concerned only with those periodic orbits that encircle the origin.

We can apply the theory of normal forms to (2.10) to make a nonlinear change of variables in (2.10) which is close to the identity and arrive at an equivalent equation

$$\begin{aligned} \dot{z}_1 &= (2\delta + \lambda\frac{4}{3})z_1 + 2\lambda z_2 - 2\beta z_2^3 - 4\beta z_1 z_2^2 \\ \dot{z}_2 &= -z_1 \end{aligned} \tag{2.11}$$

up through terms of order $(\delta+\lambda)^2|z|+|z|^4$.

The complete unfolding of the singularity in (2.11) is known (see Takens (1974a), (1974b), Carr (1981)). The case $\beta>0$ has a simple structure for the periodic orbits (they all encircle the origin) whereas the case $\beta<0$ is very complicated (periodic orbits for some values of the parameters encircle points other than the origin). As remarked earlier, we are interested only in those that encircle the origin. This means that the unfolding theory does not help in this problem. However, the techniques used in this theory can be adapted to discuss the above problem and to complete the proof of Theorem 1.1. As we will see, the computations are very complicated since the expression ϵ, which determines the period of the solutions of (1.4), involves Abelian integrals.

Let us be a little more specific about how this is done. We consider only the case where $\beta<0$ since this is by far the most complicated. To analyze the periodic solutions of (2.11), it is convenient to rescale variables

$$\lambda=-\alpha\mu^2,\ h=\mu\delta,\ u_1(t)=\frac{\sqrt{-\beta}}{\mu}z_2(-\frac{t}{\sqrt{2}\mu}),\ u_2(t)=\frac{\sqrt{-\beta}}{\sqrt{2}\mu^2}z_1(-\frac{t}{\sqrt{2}\mu}),$$

in (2.11) to obtain the equivalent system

$$\begin{aligned}\dot{u}_1&=u_2\\ \dot{u}_2&=\alpha u_1+\gamma u_2-u_1^3-2\sqrt{2}\mu u_2u_1^2+O(\mu^2[u_1+u_2]^3),\end{aligned}\tag{2.12}$$

where $\gamma=-\sqrt{2}(\delta-\frac{2}{3}\alpha\mu)$. For $\mu=0$, $\delta=0$, we obtain the conservative system

$$\begin{aligned}\dot{u}_1&=u_2\\ \dot{u}_2&=\alpha u_1-u_1^3\end{aligned}\tag{2.13}$$

with the first integral

$$H(\alpha,u_1,u_2)=\frac{u_2^2}{2}-\frac{\alpha u_1^2}{2}+\frac{u_1^4}{4},\tag{2.14}$$

Equation (2.12) has the same form as Equation (4.4.1) in Carr (1981, p.64) and so we can use his technique with the first integral (2.14) to find conditions on μ,δ to ensure that there are periodic solutions.

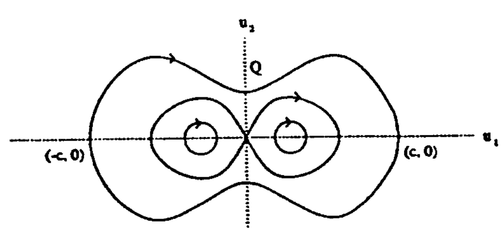

Figure 2.1. Phase portrait for the conservative system for $\alpha>0$.

The phase portrait for the Hamiltonian system (2.13) is shown in Figure 2.1 for $\alpha > 0$. There are periodic orbits encircling the equilibrium points $(\pm\sqrt{\alpha}, 0)$ However, these are of no interest to us since they do not encircle the origin. There are two homoclinic orbits to the origin which form a figure eight. There also are periodic orbits encircling the figure eight and these are of interest to us.

If $\alpha < 0$ there is only one equilibrium point of (2.13) and all orbits are periodic and encircle the origin. We are interested in all of these.

To understand the effect of the perturbation terms in (2.12), we compute the derivative of the function $H(\alpha, u_1, u_2)$ along the solutions of (2.12) to obtain

$$\dot{H}(\alpha, u_1, u_2) = \gamma u_2^2 - 2\sqrt{2}\mu u_1^2 u_2^2 + \dots, \tag{2.15}$$

where ... represents higher order terms. We want to find a periodic solution near to the periodic solution of the conservative system that passes through the point $(c, 0)$, $c > 0$ or equivalently through the point $Q \equiv (0, (\frac{c^4}{2} - \alpha c^2)^{1/2})$ on the u_2-axis (see Figure 2.1). If $u(t, Q) = (u_1(t, Q), u_2(Q))$ is the solution of (2.12) through the point Q, then, for μ and δ sufficiently small, there are constants $t_-^* < 0 < t_+^*$ such that $u_2(t_-^*, Q) = 0, u_2(t_+^*, Q) = 0$. It is now possible to show that the symmetry in (2.3) implies that $u(t, Q)$ is periodic if and only if $H(\alpha, u_1(t_-^*, Q), 0) = H(\alpha, u_1(t_+^*, Q), 0)$. If we perform an integration in (2.15), we deduce from this remark that

$$-\sqrt{2}(\delta - \frac{2}{3}\alpha\mu) = 2\sqrt{2}P(\alpha, c)\mu + \dots, \tag{2.16}$$

where ... denotes higher order terms and

$$\begin{aligned} P(\alpha, c) &= \frac{J_1(\alpha, c)}{J_0(\alpha, c)} \\ J_1(\alpha, c) &= \int_0^c w^2(\alpha w^2 - \frac{w^4}{2} + \frac{c^4}{2} - \alpha c^2)^{1/2}\, dw \\ J_0(\alpha, c) &= \int_0^c (\alpha w^2 - \frac{w^4}{2} + \frac{c^4}{2} - \alpha c^2)^{1/2}\, dw, \end{aligned} \tag{2.17}$$

where $c > \sqrt{2\alpha}$ if $\alpha \geq 0$ and $c > 0$ if $\alpha \leq 0$.

For $\bar{\alpha} > 0, \bar{c} > \sqrt{2\bar{\alpha}}$, let

$$I(\bar{\alpha}, \bar{c}) = \{(\alpha, c) : \alpha \in [-\bar{\alpha}, \bar{\alpha}], \bar{c} \geq c > \sqrt{2\alpha} \text{ if } \alpha > 0,\ \bar{c} \geq c > 0 \text{ if } \alpha \leq 0\}.$$

We can now apply the Implicit Function Theorem to (2.16) to obtain the following result.

Lemma 2.1. *For each fixed* $\bar{\alpha} > 0$, $\bar{c} > \sqrt{2\bar{\alpha}}$, *there are a constant* $\bar{\mu} > 0$ *and a function*

$$\Gamma(\alpha,\, c,\, \mu) = 2\sqrt{2}P(\alpha,\, c)\mu + O(\mu^2),$$

defined for $0 \le \mu \le \bar{\mu}$, $(\alpha, c) \in I(\bar{\alpha}, \bar{c})$, *such that (2.12) has a unique periodic solution passing through* $u_1 = 0$, $u_2 = [c^4/2 - \alpha c^2]^{1/2}$ *if and only if*

$$-\sqrt{2}(\delta - \frac{2}{3}\alpha\mu) = \Gamma(\alpha,\, c,\, \mu) = 2\sqrt{2}P(\alpha,\, c)\mu + O(\mu^2)\,. \tag{2.18}$$

Furthermore, if $\alpha > 0$, *then the periodic solution tends to the pair of homoclinic orbits (the figure eight) of (2.12) as* $c \to \sqrt{2\alpha}$ *and, if* $\alpha \le 0$, *the periodic solution tends to zero as* $c \to 0$.

Formula (2.18) determines δ as a function $\delta(\mu)$ of μ so that there is a periodic solution of (2.12) with initial data $Q = Q(c) = (0,\, [c^4/2 - \alpha c^2]^{1/2})$. Let $\delta = \delta(\mu)$ and let $u(t) = (u_1(t,\, \mu,\, \alpha,\, c),\, u_2(t,\, \mu,\, \alpha,\, c))$ (resp., $u_0(t) = (u_{10}(t,\, \alpha,\, c),\, u_{20}(t,\, \alpha,\, c))$) be the periodic solution of (2.12) (resp., of (2.13)) of period $T(\mu,\, \alpha,\, c)$ (resp., $T_0(\alpha,\, c)$) passing through $Q = (0,\, [c^4/2 - \alpha c^2]^{1/2})$. Since $u(t,\, \mu,\, \alpha,\, c) \to u_0(t,\, \alpha,\, c)$ as $\mu \to 0$, we have $T(\mu,\, \alpha,\, c) \to T_0(\alpha,\, c)$ as $\mu \to 0$. Furthermore,

$$T(\mu,\, \alpha,\, c) = T_0(\alpha,\, c) + O(\mu)\,. \tag{2.19}$$

From the above scaling, this periodic solution leads to a periodic solution of (2.11) of period $T(\mu,\, \alpha,\, c)/\sqrt{2}\mu$, which in turn gives a periodic solution of (2.4) of period $2 + 2r\epsilon$ if and only if $2 + 2r\epsilon = \epsilon r(T(\mu,\, \alpha,\, c)/\sqrt{2}\mu)$, $\lambda = -\mu^2$ and $r = 1 + \mu\delta$, where δ is given by (2.18). Therefore,

$$\epsilon = \epsilon(\mu,\, \alpha,\, c) = \frac{2\sqrt{2}\mu}{[T(\mu,\, \alpha,\, c) - 2\sqrt{2}\mu][1 + \mu^2(\frac{2\alpha}{3} - 2P(\alpha,\, c))]}\,. \tag{2.20}$$

Here, we omit the terms of order $O(\mu^3)$ since they play no essential role.

Lemma 2.2. *Let* $\bar{c} > \sqrt{2\bar{\alpha}}$ *be given, let* $\bar{\mu}$ *be defined as in Lemma 2.1 and let* $\epsilon(\mu,\, \alpha,\, c)$ *be given by (2.20). Then, for each fixed* $\mu \in (0,\, \bar{\mu}]$ *and* $\alpha \in [0,\, \bar{\alpha}]$, *the function* $\epsilon(\mu,\, \alpha,\, c)$ *is monotone increasing for* $c \in (\sqrt{2\alpha},\, \bar{c}]$ *provided that* $\epsilon(\mu,\, \alpha,\, c) > 0$. *In addition,*

$$\lim_{c \to \sqrt{2\alpha}} \epsilon(\mu,\, \alpha,\, c) = 0\,. \tag{2.21}$$

We remark that the periodic orbit $\{\, u(t,\, \mu,\, \alpha,\, c)\,, t \in \mathbb{R}\,\}$ tends to a pair of homoclinic orbits (the figure eight of (2.12)) as $c \to \sqrt{2\alpha}$. Therefore, $T(\mu,\, \alpha,\, c) \to \infty$ as $c \to \sqrt{2\alpha}$, which implies the relation (2.21).

The proof of this lemma is the most difficult part of the proof because of the difficulties in understanding how ratios of the Abelian integrals in (2.17) depend upon the parameter c. We refer to Hale and Huang (1992a) for the details.

Lemma 2.2 completes the proof of Theorem 3.1.

3. Hybrid systems. In applications, we often encounter equations which are generalizations of (3.1) and consist of differential difference equations coupled with difference equations. We refer to such systems as ***hybrid systems*** and the form that we consider is the following:

$$\begin{aligned} \epsilon\dot{x}(t) &= f_\lambda(x(t), y(t)) \\ y(t) &= g_\lambda(x(t), y(t), x(t-1), y(t-1)), \end{aligned} \tag{3.1}$$

where $\epsilon > 0$, λ are small real parameters, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$ are vectors and the functions $f_\lambda(x, y) = f(x, y, \lambda)$ and $g_\lambda(x, y, z, w) = g(x, y, z, w, \lambda)$ are smooth vector valued functions which vanish for all variables equal to zero.

This system includes (1.4) by letting the function g_λ be $x(t-1)$ and the function f_λ be replaced by $-x(t) + f_\lambda(y(t))$. System (3.1) includes also

$$\epsilon\dot{x}(t) + x(t) = f_\lambda(x(t-1), x(t-2)) \tag{3.2}$$

by letting y be a two vector $y_1(t) = x(t-1), y_2(t) = y_1(t-1) = x(t-2)$. In this way, we are able to consider situations in which the map defined by $\epsilon = 0$ is a two dimensional map; for example, the Henon map.

In models of transmitted light through ring cavities with several chambers, Vallee, Dubois, Coté and Delisle (1987), Vallee and Marriott (1989), have proposed the following model:

$$(\delta_m \frac{d}{dt} + 1) \cdots (\delta_1 \frac{d}{dt} + 1) z(t) = h_\lambda(z(t-1)), \tag{3.3}$$

where each $\delta_j > 0$ is a small parameter and z is a scalar.

We can transform this to an equivalent matrix system

$$\begin{aligned} \delta_1\dot{x}_1(t) + x_1(t) &= x_2(t) \\ &\cdots\cdots \\ \delta_{m-1}\dot{x}_{m-1}(t) + x_{m-1}(t) &= x_m(t) \\ \delta_m\dot{x}_m(t) + x_m(t) &= h_\lambda(x_1(t-1)). \end{aligned} \tag{3.4}$$

If we assume that δ_j as $\delta_j = \epsilon\alpha_j^{-1}$, $j = 1, 2, \ldots, m$, where each $\alpha_j > 0, j = 1, 2, \ldots, m$, then we obtain the matrix equation

$$\epsilon\dot{x}(t) + Ax(t) = Af_\lambda(x(t-1)), \tag{3.5}$$

where A and f_λ are given by

$$(3.6) \qquad A = \begin{bmatrix} \alpha_1 & -\alpha_1 & 0 & \dots & 0 & 0 \\ 0 & \alpha_2 & -\alpha_2 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 & \dots & \alpha_{m-1} & -\alpha_{m-1} \\ 0 & 0 & 0 & \dots & 0 & \alpha_m \end{bmatrix} \qquad f_\lambda(x) = \begin{bmatrix} h_\lambda(x_1) \\ h_\lambda(x_1) \\ \cdot \\ \cdot \\ \cdot \\ h_\lambda(x_1) \\ h_\lambda(x_1) \end{bmatrix}$$

System (3.5) is a special case of (3.1) if we put $y(t) = x(t-1)$.

It is of interest to consider (3.5) for a general function f_λ and a general matrix A, assuming only that A^{-1} exists. Such a system is a special case of the system

$$(3.7) \qquad \begin{aligned} \epsilon\dot{x}(t) + Ax(t) &= Af_\lambda(y(t)) \\ y(t) &= g_\lambda(x(t-1),\, y(t-1))\,, \end{aligned}$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$ are vectors, the $m \times m$ matrix A has an inverse and the functions $f_\lambda(y)$ and $g_\lambda(x,\, y)$ are smooth vector valued functions. Equation (3.7) has been used by Ikeda (1979), Ikeda, Daido and Akimoto (1980) as a model of a ring cavity containing a nonlinear dielectric medium for which a part of the transmitted light is fed back into the medium.

The equation (3.1) arises also in the theory of transmission lines. If the lines are lossless and described by the telegraph equations with the boundary conditions for the circuitry between the lines reflecting Kirchoff's laws, it has been known for a long time that the flow can be described by an equivalent set of neutral delay differential equations (see, for example, Hale and Verduyn-Lunel (1993) for a discussion and references). Many of these same problems also can be written in the form (3.1). For example, by an appropriate change of coordinates, the equation studied by Shimura (1967) for a transmission line with a tunnel diode and a lumped parallel capacitance can be rewritten as

$$(3.8) \qquad \begin{aligned} \epsilon\dot{x}(t) &= y(t) - g(x(t)) \\ y(t) &= \alpha + Ky(t-1) - x(t) - Lx(t-1)\,, \end{aligned}$$

where $(x(t), y(t))$ represent the voltage and current at one end of the line, all constants are positive and represent physical parameters. Under reasonable assumptions in the model, the parameter ϵ can be considered to be very small. In the paper of Shimura (1967), several wave forms were observed numerically which compared reasonably well with experimental results. Some of these wave forms were very similar to square waves.

As remarked before, in many situations, for fixed $\epsilon_0 > 0$, there is a $\lambda^*(\epsilon_0)$ (which we assume to be positive for definiteness) such that (3.1) undergoes a generic Hopf bifurcation at $(\epsilon, \lambda, x, y) = (\epsilon_0, \lambda^*(\epsilon_0), 0, 0)$ to a periodic solution $(x^*_{\epsilon\lambda}, y^*_{\epsilon\lambda})$. Let us assume that there is a neighborhood U of $(\epsilon, \lambda) = (0, 0)$ such that $(x^*_{\epsilon\lambda}, y^*_{\epsilon\lambda})$ exists for all $(\epsilon > 0, \lambda)$ in U. Our objective remains to understand the behavior of the profile of $(x^*_{\epsilon\lambda}, y^*_{\epsilon\lambda})$ as $\epsilon \to 0$. We are not able to do this in the general context described, but we can say something if we impose more conditions on the functions f_λ, g_λ.

It is to be expected that the limiting profile is in some way related to the equation obtained by putting $\epsilon = 0$ in (3.1). For $\epsilon = 0$, we suppose that the resulting equation (3.1) defines a map on $\mathbb{R}^m \times \mathbb{R}^n$:

$$(x, y) \in \mathbb{R}^m \times \mathbb{R}^n \mapsto T_\lambda(x, y) \in \mathbb{R}^m \times \mathbb{R}^n \tag{3.9}$$

for which the origin is stable for $\lambda < 0$ and unstable for $\lambda > 0$. Let us also suppose that $T_\lambda(x, y)$ undergoes a generic period doubling bifurcation at $(x, y, \lambda) = (0, 0, 0)$ with the period two points being $d_{j\lambda} \in \mathbb{R}^m \times \mathbb{R}^n$, $j = 1, 2$. If the bifurcation is supercritical, we can define the square wave 2-periodic function and, if the bifurcation is subcritical, we can define the pulse wave 2-periodic function as in Section 1.

Under some conditions on the functions f_λ, g_λ, the generic period doubling bifurcation of $T_\lambda(x, y)$ leads to a generic Hopf bifurcation in (3.1) which is supercritical (resp. subcritical) if the period doubling bifurcation is supercritical (resp. subcritical). The natural question as before is the following:

Is it possible that the limiting profile of the periodic solution $(x^*_{\epsilon\lambda}, y^*_{\epsilon\lambda})$ obtained through the Hopf bifurcation is either the square wave or pulse wave?

In a later section, we present general results of Hale and Huang (1994) for which this is true for equation (3.7). Preliminary computations indicate that similar results also hold for systems of the form (3.8).

4. Hopf bifurcation in a special hybrid system. In this section, we give conditions under which there will be a generic first Hopf bifurcation for the hybrid system

$$\begin{aligned} \epsilon\dot{x}(t) + Ax(t) &= Af_\lambda(y(t)) \\ y(t) &= g_\lambda(x(t-1), y(t-1)), \end{aligned} \tag{4.1}$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$ are vectors, the $m \times m$ matrix A has an inverse and the functions $f_\lambda(y)$ and $g_\lambda(x, y)$ are smooth vector valued functions.

We make the following hypothesis:

$$A^{-1} \text{ exists}, \quad f_\lambda(0) = 0, \quad g_\lambda(0, 0) = 0. \tag{H1}$$

(H1) is natural since we want to consider bifurcation from the origin and we also want a map to be defined for $\epsilon = 0$.

If we introduce the notation,

$$
\begin{aligned}
&A_2(\lambda) = D_y f_\lambda(0), \quad B_1(\lambda) = D_x g_\lambda(0,0), \quad B_2(\lambda) = D_y g_\lambda(0,0) \\
&C(\lambda) \equiv B_1(\lambda) A_2(\lambda) + B_2(\lambda),
\end{aligned}
\tag{4.2}
$$

then the linear variational equation of (4.1) about the origin is

$$
\begin{aligned}
\epsilon \dot{x}(t) + Ax(t) &= AA_2(\lambda) y(t) \\
y(t) &= B_1(\lambda) x(t-1) + B_2(\lambda) y(t-1),
\end{aligned}
\tag{4.3}
$$

for which the characteristic matrix is

$$
\Delta(\lambda, \epsilon, \mu) = \begin{bmatrix} \epsilon\mu I_m + A & -AA_2(\lambda) \\ -B_1(\lambda)e^{-\mu} & I_n - B_2(\lambda)e^{-\mu} \end{bmatrix}.
\tag{4.4}
$$

We let $\sigma(C)$ denote the spectrum of a square matrix C and let $B_\rho = \{z \in \mathbb{C} : |z| \leq \rho\}$. Our next hypothesis is

$$
\sigma(B_2(0)) \subset B_\rho, \quad \rho < 1.
\tag{H2}
$$

Without hypothesis (H2), it is not too difficult to show that there would be solutions of (4.4) which either accumulate or lie on a line in the complex plane with real part ≥ 0. If this were the case, we would not be able to find a simple Hopf bifurcation.

It is convenient to introduce the following definition. We say that a curve Γ in the (λ, ϵ)-plane, $\epsilon > 0$, is a *Hopf Bifurcation Curve* of (4.3) if there is an $\epsilon^* > 0$ and a continuous function $\lambda = \lambda(\epsilon), 0 < \epsilon \leq \epsilon^*$, $\lambda(\epsilon) \to 0$ as $\epsilon \to 0$ such that, if $\Gamma_{\epsilon^*} = \{(\lambda, \epsilon) : \lambda = \lambda(\epsilon), \epsilon \in (0, \epsilon^*]\}$, then, for any $(\lambda_0, \epsilon_0) \in \Gamma_{\epsilon^*}$, there are two purely imaginary solutions $\pm i\beta_0$ of the characteristic equation $\det \Delta(\lambda_0, \epsilon_0, \mu) = 0$ and the remaining solutions μ satisfy $\operatorname{Re} \mu \neq 0$. We say that a Hopf Bifurcation Curve is a *Generic Hopf Bifurcation Curve with respect to* ϵ if, for fixed λ_0, the two eigenvalues $\mu(\lambda_0, \epsilon), \bar{\mu}(\lambda_0, \epsilon)$, $\mu(\lambda_0, \epsilon_0) = i\beta_0$, satisfy $\operatorname{Re} \partial\mu(\lambda_0, \epsilon_0)/\partial\epsilon < 0$. This type of transversal crossing of the imaginary axis of the eigenvalue $\mu(\lambda_0, \epsilon)$ implies that there will be a Hopf bifurcation with respect to ϵ at ϵ_0. We say that a Hopf Bifurcation Curve is the *First Hopf Bifurcation Curve* if, for each $(\lambda_0, \epsilon_0) \in \Gamma_{\epsilon^*} j$, all eigenvalues μ corresponding to the parameters $\epsilon > \epsilon_0$, $\lambda = \lambda_0$, have $\operatorname{Re} < 0$. The *Generic First Hopf Bifurcation Curve with respect to* ϵ is a Generic Hopf Bifurcation Curve and also a First Hopf Bifurcation Curve. The Generic First Hopf Bifurcation Curve with respect to ϵ is the most interesting because there is a transfer of stability of the origin at $\epsilon = \epsilon_0$; that is, the origin is stable for $\epsilon > \epsilon_0$ and unstable for $\epsilon < \epsilon_0$. From the physical origins of the problem, this is natural because we expect that the origin is stable for large ϵ (by a change of time scale, this is small delay) and to eventually become unstable for small ϵ (large delay). The Generic First Hopf Bifurcation Curve with respect to ϵ represents the first change in the stability properties of the origin.

Huang and Hale (1994a) have given conditions on the coefficients in (4.2) for the existence of the Generic First Hopf Bifurcation Curve.

In addition to (**H2**), assume that

$$\textbf{(H3)} \qquad \begin{aligned} &-(1+\lambda) \in \sigma(C(\lambda)) \text{ is a simple eigenvalue,} \\ &\sigma(C(\lambda)) \setminus \{-(1+\lambda)\} \subset B_\rho \text{ with } \rho < 1. \end{aligned}$$

If (**H3**) is satisfied, then we can introduce a change of variables in y to obtain

$$C(\lambda) = \begin{bmatrix} -(1+\lambda) & 0 \\ 0 & H_0(\lambda) \end{bmatrix}, \quad \sigma(H_0(\lambda) \subset B_\rho, \ \ \rho < 1. \tag{4.5}$$

We also introduce the notation

$$-B_1(0)A^{-1}A_2(0) \equiv S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \quad -B_1(0)A^{-2}A_2(0) \equiv W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}. \tag{4.6}$$

The remaining hypotheses that we impose are

$$\textbf{(H4)} \qquad R_0 \equiv -S_{11}^2 + 2[W_{11} + S_{12}(I + H_0(0))^{-1}S_{21}] \neq 0, \quad S_{11} \neq 0.$$

$$\textbf{(H5)} \qquad \min\{\operatorname{Re} z : z \in \sigma(A)\} > 0,$$

$$\textbf{(H6)} \qquad \operatorname{Det} \begin{bmatrix} i\theta I_m + A & -AA_2(0) \\ -B_1(0)e^{-iv} & I_n - B_2(0)e^{-iv} \end{bmatrix} \neq 0, \text{ for } \theta > 0, 0 \leq v \leq 2\pi, (\theta, v) \neq (0, \pi)$$

$$\textbf{(H7)} \qquad \operatorname{Det} \begin{bmatrix} \mu I_m + A & -AA_2(0) \\ -B_1(0) & I_n - B_2(0) \end{bmatrix} \neq 0, \text{ for all } \mu \in \mathbb{C}, \operatorname{Re}\mu \geq 0.$$

One of the main results of this section is

Theorem 4.1. *Under the assumptions* (**H1**)*-*(**H5**)*, the hypotheses* (**H6**) *and* (**H7**) *are necessary and sufficient for the existence of the Generic First Hopf Bifurcation Curve with respect to* ϵ*. Furthermore, if these hypotheses are satisfied, then there is an* $\epsilon^* > 0$ *such that this curve is given as* $\Gamma_{\epsilon^*} = \{((\lambda(\epsilon), \epsilon), \epsilon \in (0, \epsilon^*\}$*, where* $\lambda(\epsilon)$ *is a* C^2*-function having the property that* $\lambda(\epsilon) \to 0$ *and is given approximately by*

$$\lambda(\epsilon) = \frac{1}{2}\pi^2 R_0 \epsilon^2 + o(\epsilon^2) \tag{4.7}$$

as $\epsilon \longrightarrow$ *zero.*

Formula (4.7) implies that the Generic First Hopf Bifurcation Curve with respect to ϵ in the (λ, ϵ)-plane is the graph over the positive (resp. negative) λ-axis if $R_0 > 0$ (resp. $R_0 < 0$).

We do not prove this theorem and refer the reader to Hale and Huang (1994b).

Let us see if the hypotheses of Theorem 4.1 are satisfied for some of our examples. We have remarked before that (1.4) is a special case of (4.1) when it is written as

$$\begin{aligned} \epsilon\dot{x}(t) + x(t) &= f_\lambda(y(t)) \\ y(t) &= x(t-1). \end{aligned} \tag{4.8}$$

Let us assume that

$$f_\lambda(x) = -(1+\lambda)x + ax^2 + bx^3, \tag{4.9}$$

where $\beta \equiv a^2 + b \neq 0$. The linear variational equation about $x = 0$ for (4.8) is a special case of (4.3) with $A = 1$, $A_2(\lambda) = -(1+\lambda)$, $B_1(\lambda) = 1$, $B_2(\lambda) = 0$. It is now obvious that the hypotheses (H1)-(H5) are satisfied. A simple computation shows that hypotheses (H6) is equivalent to $i(\theta - \sin v) + 1 + \cos v \neq 0$ for $\theta > 0$, $0 \leq v \leq 2\pi$, $v \neq \pi$. This is clearly satisfied. Also, (H7) is equivalent to $\mu + 2 \neq 0$ for Re $\mu \geq 0$, which is true. Therefore, there is a Generic First Hopf Bifurcation Curve with respect to ϵ.

Let us next discuss equation (3.5), (3.6). We show that all of the hypotheses are satisfied by analyzing directly the characteristic equation. The proof is not difficult. The characteristic equation for the linearization about $x = 0$ is given by

$$E(\epsilon, \lambda, \mu) \equiv (\epsilon\alpha_m\mu + 1)\cdots(\epsilon\alpha_1\mu + 1) + (1+\lambda)e^{-\mu} = 0. \tag{4.10}$$

In this particular situation, the determinant in (H6) is given by

$$E_1(\theta, v) \equiv E(\frac{\theta}{v}, 0, iv) = \prod_{j=1}^{m}(i\theta\alpha_j + 1) + e^{-iv}.$$

If $\theta \geq 0, 0 \leq v < 2\pi$, are such that $E_1(\theta, v) = 0$, then we have $|\prod_{j=1}^{m}(i\theta\alpha_j + 1)| = 1$; that is, $\prod_{j=1}^{m}(\theta^2\alpha_j^2 + 1) = 1$. Since $\alpha_j > 0$, we must have $\theta = 0$ and hence $1 + e^{-iv} = 0$; that is, $v = \pi$. Therefore, the condition (H6) is satisfied.

Next, the function in (H7) has the form

$$C_2(\mu) \equiv \prod_{j=1}^{m}(\mu\alpha_j + 1) + 1.$$

If there is a $\mu = u + iv, u \geq 0$, such that $C_2(u + iv) = 0$, then it follows that $1 = \prod_{j=1}^{m}((u\alpha_j + 1)^2 + v^2)$. Therefore, we must have $u = v = 0$, which leads to the assertion that $0 = C_2(0) = 1 + 1$, which is a contradiction. As a consequence, $C_2(\mu) \neq 0$ for all $\mu \in \mathbb{C}$ with Re $\mu \geq 0$; that is, (H7) is satisfied.

We can now show that, for fixed $\epsilon_0 > 0$, there is a unique $\lambda^*(\epsilon_0) > 0$ such that (4.10) has exactly two purely imaginary roots and the remaining ones have negative real parts for $(\lambda, \epsilon) = (\lambda(\epsilon_0), \epsilon_0)$. For $\epsilon > \epsilon_0$, the origin is asymptotically stable and, for $0 < \epsilon < \epsilon_0$, the origin is unstable. In this way, we obtain the existence of a Generic First Hopf Bifurcation Curve with respect to ϵ.

After we know that a Generic First Hopf Bifurcation Curve with respect to ϵ exists, the next step is to determine the direction of bifurcation for the nonlinear equation. To describe the result, we need some additional notation. It turns out that the direction of bifurcation is related to the map

$$y \in \mathbb{R}^n \mapsto \mathcal{F}_\lambda(y) \equiv g_\lambda(f_\lambda(y), y) \in \mathbb{R}^n \tag{4.12}$$

obtained from (4.1) for $\epsilon = 0$. If we assume that $C(\lambda)$ is given as in (4.5) and let $y = \text{col}\,(y_1, y_2) \in \mathbb{R}^1 \times \mathbb{R}^{n-1}$, $\mathcal{F}_\lambda = \text{col}\,(\mathcal{F}_{1\lambda}, \mathcal{F}_{2\lambda}) \in \mathbb{R}^1 \times \mathbb{R}^{n-1}$, then we can write

$$\begin{aligned} \mathcal{F}_{1\lambda}(y) &= -(1+\lambda)y_1 + k_1(\lambda)y_1^2 + y_1 k_2(\lambda)y_2 + k_3(\lambda)y_1^3 + O(\|y_2\|^2 + y_1^2\|y_2\| + y_1^4) \\ \mathcal{F}_{2\lambda}(y) &= H_0(\lambda)y_2 + y_1^2 H_1(\lambda) + y_1 H_2(\lambda)y_2 + O(\|y_2\|^2 + y_1^2\|y_2\| + \|y\|^3) \end{aligned} \tag{4.11}$$

as $(y, \lambda) \to (0, 0)$.

We assume that

$$R_1 \equiv k_2(0)[I - H_0(0)]^{-1}H_1(0) + k_1^2(0) + k_3(0) \neq 0. \tag{H8}$$

The following result is due to Hale and Huang (1994b).

Theorem 4.2. *Suppose that (H1)-(H8) are satisfied and Γ_{ϵ^*} is the Generic First Hopf Bifurcation Curve with respect to ϵ given in Theorem 4.1. If, in addition, $R_0 > 0$, then, for each fixed $\lambda_0 = \lambda(\epsilon_0)$, ϵ_0 sufficiently small, system (4.4) has a generic Hopf bifurcation from the origin with respect to ϵ at ϵ^* which is supercritical (resp. subcritical) if $R_1 > 0$ (resp. $R_1 < 0$).*

The proof of this is very long and uses results from a very general Hopf bifurcation theorem of Hale and Huang (1994a). The idea for the proof is simple and standard. It is first observed that $R_0 > 0$ implies that the origin is stable for $\epsilon > \epsilon^*$ and unstable for $\epsilon < \epsilon^*$. From the physical point of view, this is to be expected and therefore the assumption that $R_0 > 0$ is natural. Also, there is a bifurcation function $G(a, \epsilon)$ for periodic solutions near the origin, where a is approximately the amplitude of the periodic solution. There is a periodic solution of (4.1) near the origin if and only if there are (a, ϵ) such that $G(a, \epsilon) = 0$. This function is odd in a and it is shown that the linear term in a has the same sign as $-R_0$. Furthermore, the cubic term in a has the same sign as ϵR_1. Therefore, there is at most one periodic solution and it satisfies the properties stated in the theorem.

The following result is in Hale and Huang (1994b) and is a standard application of the method of Liapunov-Schmidt.

Theorem 4.3. *If* **(H1)**, **(H3)** *and* **(H8)** *are satisfied, then there is a generic period two doubling bifurcation of $\mathcal{F}_\lambda$ at $(y, \lambda) = (0, 0)$. More precisely, if $R_1\lambda > 0$, then there are period two points $d_{1\lambda}, d_{2\lambda}$ of $\mathcal{F}_\lambda$ such that $\mathcal{F}_\lambda(d_{1\lambda}) = d_{2\lambda}, \mathcal{F}_\lambda(d_{2\lambda}) = d_{1\lambda}$, which is stable for $R_1 > 0$ (supercritical bifurcation) and unstable for $R_1 < 0$ (subcritical bifurcation).*

This is a very interesting result because it says that, if we have $R_0 > 0$ and the conditions in Theorem 4.2 for Generic First Hopf Bifurcation with respect to ϵ to occur, then also there is generic period doubling of the map with respect to λ and the direction of bifurcation of the Hopf bifurcation with respect to ϵ is the same as the direction of bifurcation of the period doubling bifurcation of the map with respect to λ.

We remark that the converse of this statement may not be true. More precisely, it is possible for the map in (4.12) to generically period double at $\lambda = 0$ and yet the linearization about the origin in the differential euation may not possess a Generic First Hopf Bifurcation Curve with respect to ϵ.

5. Square and pulse waves for hybrid systems. It remains to show that the periodic orbit obtained from the Hopf bifurcation can be extended away form the Hopf bifurcation curve, is unique and has limiting values related to either the square or pulse periodic functions related to the period two points of the map. We want to apply a method similar to the one used in the proof of Theorem 1.1; namely scaling, center manifolds, etc.

As in Section 2, we seek periodic solutons of (4.1) with a period $2 + 2(r_0 + h)\epsilon$, where r_0 is a fixed parameter (to be determined later) which depends only upon the matrices $A, A_2(0), B_1(0), B_2(0)$ and h will be determined as a function of ϵ. As in Section 2, if $(x(t), y(t))$ is such a solution of (4.1), we introduce the transformation

$$
\begin{aligned}
u_1(t) &= x(-\epsilon(r_0 + h)t), \quad u_2(t) = x(-\epsilon(r_0 + h)t + 1 + \epsilon(r_0 + h)) \\
v_1(t) &= y(-\epsilon(r_0 + h)t), \quad v_2(t) = y(-\epsilon(r_0 + h)t + 1 + \epsilon(r_0 + h)).
\end{aligned}
\tag{5.1}
$$

Since $x(t)$ and $y(t)$ have period $2 + 2(r_0 + h)\epsilon$, we see that

$$
\begin{aligned}
u(t-1) &= x(-\epsilon(r_0 + h)t - 1) \\
v(t-1) &= y(-\epsilon(r_0 + h)t - 1).
\end{aligned}
\tag{5.2}
$$

If we use (5.1) and (5.2) in (4.1), we deduce that

$$
\begin{aligned}
\dot{u}_1(t) &= (r_0 + h)Au_1(t) - (r_0 + h)Af_\lambda(v_1(t)) \\
\dot{u}_2(t) &= (r_0 + h)Au_2(t) - (r_0 + h)Af_\lambda(v_2(t)) \\
v_1(t) &= g_\lambda(u_2(t-1), v_2(t-1)) \\
v_2(t) &= g_\lambda(u_1(t-1), v_1(t-1)).
\end{aligned}
\tag{5.3}
$$

We consider h, λ as the new bifurcation parameters. We must determine the constant r_0 and the procedure, as for the simpler case in Section 2, will be to insist that the linear variational equation of (5.3) about zero has a double eigenvalue zero with no other eigenvalues on the imaginary axis.

If we use the notation of Section 4 and let $u = \text{col}\,(u_1, u_2)$, $v = \text{col}\,(v_1, v_2)$,

$$\begin{aligned} \hat{A} &= \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad \hat{A}_2 = \begin{bmatrix} A_2(0) & 0 \\ 0 & A_2(0) \end{bmatrix}, \\ \hat{B}_j &= \begin{bmatrix} 0 & B_j(0) \\ B_j(0) & 0 \end{bmatrix}, \quad j = 1, 2, \end{aligned} \tag{5.4}$$

then the linear variational equation of (5.3) for $(h, \lambda) = (0, 0)$ is given by

$$\begin{aligned} \dot{u}(t) &= r_0 \hat{A} u(t) - r_0 \hat{A} \hat{A}_2 v(t) \\ v(t) &= \hat{B}_1 u(t-1) + \hat{B}_2 v(t-1). \end{aligned} \tag{5.5}$$

The eigenvalues of (5.5) are the solutions of the characteristic equation

$$\det \Delta(\mu, r_0) = 0, \quad \Delta(\mu, r_0) = \begin{bmatrix} \mu - r_0 \hat{A} & r_0 \hat{A} \hat{A}_2 \\ -\hat{B}_1 e^{-\mu} & I - \hat{B}_2 e^{-\mu} \end{bmatrix}. \tag{5.6}$$

Because of (2.3), (2.4) and the symmetry in (5.5), zero always is an eigenvalue. We impose conditions on the coefficients to ensure that 0 is an eigenvalue of multiplicity two and there are no other eigenvalues on the imaginary axis. To do this, we will need hypotheses (**H1**), (**H2**), (**H4**) of Section 4 as well as the following one:

$$\det \begin{bmatrix} i\omega I_{2m} - S_{11} \hat{A} & S_{11} \hat{A} \hat{A}_2 \\ -\hat{B}_1 e^{-i\omega} & I_{2n} - \hat{B}_2 e^{-i\omega} \end{bmatrix} \neq 0 \quad \text{for } \omega \in \mathbb{R} \setminus \{0\}, \tag{H9}$$

where S_{11} is given in (4.6).

The follwoing results are due to Hale and Huang (1994b).

Lemma 5.1. *If (**H1**), (**H2**), (**H4**) and (**H9**) are satisfied and $r_0 = S_{11}$, then $\mu = 0$ is an eigenvalue of (5.5) of multiplicity two and there is a $\delta > 0$ such that the remaining eigenvalues satisfy $|\mathrm{Re}\mu| \geq \delta > 0$ and there are only a finite number of eigenvalues with positive real parts.*

Theorem 5.2. *If (**H1**)-(**H4**), (**H8**)and (**H9**) are satisfied, then there is a a neighborhood U of $(0, 0)$ in the (λ, ϵ) plane and a sectorial region S in U such that, if $(\lambda, \epsilon) \in U$, then there is a periodic solution $\tilde{x}_{\lambda, \epsilon}$ of (2.4) with period $2\tau(\lambda, \epsilon) = 2 + 2S_{11}\epsilon + O(|\epsilon|(|\lambda| + |\epsilon|))$ as $(\lambda, \epsilon) \to (0, 0)$ if and only if $(\lambda, \epsilon) \in S$. Furthermore, this solution is unique.*

We do not give a proof of these results, but simply refer the readed to Hale and Huang (1994b). We only make a few remarks explaining in some detail other properties of the solutions in the sector S. Of course, the sector S must belong to the set $\epsilon > 0$ in the (λ, ϵ) plane. It actually is shown that, if $R_1 > 0$ (the supercritical case of period doubling of the map) and $R_0 > 0$, then the sector $S \subset \{(\lambda, \epsilon) : \epsilon > 0, \lambda > 0\}$ and, for $\lambda = \lambda_0 > 0$, fixed, the set $\{\epsilon : (\epsilon, \lambda_0) \in S\}$ is an interval $(0, \epsilon_0(\lambda_0))$, where

$$\epsilon_0(\lambda_0) = \frac{1}{\pi}\frac{2\pi\lambda_0}{R_0}^{\frac{1}{2}} + O(\lambda_0)$$

as $\lambda_0 \to 0$. For any $\epsilon \in (0, \epsilon_0(\lambda_0))$, the periodic solution $\tilde{x}_{\lambda_0,\epsilon}(t)$ approaches a square wave 2-periodic function as $\epsilon \to 0$.

If $R_1 < 0$ (the subcritical case of period doubling of the map), the sector S is completely different (as was the case in Theorem 1.1) and the periodic orbits have a different structure as $\epsilon \to 0$. The sector S contains points (ϵ, λ) with λ both negative and positive. More precisely, for $\lambda_0 > 0$, the set $\{\epsilon : (\epsilon, \lambda_0) \in S\}$ is an interval $(\epsilon_0(\lambda_0), \beta_0(\lambda_0))$. For $\lambda_0 < 0$, the set $\{\epsilon : (\epsilon, \lambda_0) \in S\}$ is an interval $(0, \alpha_0(\lambda_0))$. For any $\epsilon \in (0, \alpha_0(\lambda_0))$, the unique periodic solution $\tilde{x}_{\lambda_0,\epsilon}(t)$ becomes pulse like as $\epsilon \to 0$ in the following sense: the periodic solution $\tilde{x}_{\lambda_0,\epsilon}(t)$ has the property that $\tilde{x}_{\lambda_0,\epsilon}(t) \to 0$ as $\epsilon \to 0$ uniformly on compact sets of $(0, 1) \cup (1, 2)$. In the pulse like solution, the pulses in the solution occur near the integers and are opposite in direction. However, the magnitude of the pulse near the integers exceeds the magnitude of the corresponding period two point of the map.

Notice that Theorem 5.2 and Theorem 4.2 use different hypotheses. We can obtain generic first Hopf bifurcation with respect to ϵ (which imply the generic period doubling of the map) with different assumptions than for the existence of the square and pulse like solutions related to the period 2 points of the map. The differences arise in the discussion of the eigenvalues of the linear variational equation for (4.1) and the one for the scaled equations (5.3). It is possible to give an example for which the zero solution of (4.3) is stable and the zero solution of (5.5) is unstable. Naturally, this leaves open some questions as to whether we have obtained the best possible results.

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Part 3. Small Delays Can Make a Difference

1. Difference Equations. In Part 2, we have discussed in some detail the effects of large delays on the dynamics of retarded delay differential equations. For the problems considered, we have seen that, under reasonable assumptions, the limiting dynamics as the delay approached infinity presented no surprises, at least locally near an equilibrium point. The limiting dynamics was determined by the local dynamics of a map.

If we consider a retarded delay differential equation (or even more generally a functional differential equation) for which the delays are small, then it is possible to prove that the limiting dynamics is determined by the ordinary differential equation obtained by putting all of the delays equal to zero (see Kurzweil (1970), (1971) and a more complete discussion in Hale, Magalhães and Oliva (1984)). In such a situation, it is fair to say that small delays are unimportant. For retarded functional differential equations, the same general remarks are valid if we make small changes in the delays with the change not necessarily occuring around zero.

In this section, we devote our attention to similar problems for neutral differential difference equations. These are equations for which the derivative of the solution also occurs with a delay. Special cases of such equations of course would by difference equatons since we can differentiate to obtain a neutral differential difference equation. In the next section, we will see also that these problems are very closely related to the control of hyperbolic PDE when the control function is applied on the boundary with a time delay.

For any operator T on a Banach space, we let $\sigma(T)$ denote the spectrum of T, $r(\sigma(t))$ denote the radius of the spectrum and $r_e(\sigma(T))$ denote the radius of the essential spectrum. We begin with an example of a difference equation.

Example 1.1 (*Uniform asymptotic stability may be destroyed by small changes in the delay*). The observations in this example are essentially due to Melvin (1974). Suppose that $c_0 > 0$, $0 < r_1 \leq r_2 \leq c_0$, are constants and consider the difference equation

$$x(t) + \frac{1}{2}x(t-r_1) + \frac{1}{2}x(t-r_2) = 0. \tag{1.1}$$

Let $C = C([-c_0, 0]; \mathbb{R})$ and let

$$C_0 \equiv C_{0,(r_1,r_2)} = \{\varphi \in C : \varphi(0) + \frac{1}{2}\varphi(-r_1) + \frac{1}{2}\varphi(-r_2) = 0\}.$$

For any $\varphi \in C_0$, there is a unique solution $x(t,\varphi)$ of (1.1) which is defined for all $t \in \mathbb{R}$. If we define $[S_{(r_1,r_2)}(t)\varphi](\theta) = x(t+\theta,\varphi)$, $\theta \in [-c_0, 0]$, then $S_{(r_1,r_2)}(t)$, $t \geq 0$, is a C^0-semigroup on $C_{0,(r_1,r_2)}$.

We need the following well known result (see, for example, Hale and Verduyn-Lunel (1993)). The proof is nontrivial and makes extensive use of properties of the Laplace transform.

Theorem 1.1. *$r(\sigma(S_{(r_1,r_2)}(1)) = e^{\alpha}$, where α is defined by*

$$\alpha \equiv \alpha(r_1, r_2) \equiv \sup\{\operatorname{Re}\lambda : \Delta_{0,(r_1,r_2)}(\lambda) = 0\}, \tag{1.2}$$

$$\Delta_{0,(r_1,r_2)}(\lambda) = 1 + \frac{1}{2}e^{-\lambda r_1} + \frac{1}{2}e^{-\lambda r_2}.$$

We now compute $r(\sigma(S_{(r_1,r_2)}(1))$ for some particular values of (r_1, r_2). For $(r_1, r_2) = (1, 2)$, $\Delta_{0,(1,2)}(\lambda)$ is a quadratic function in $e^{-\lambda}$ and it is easy to show that the solutions of (1.2) satisfy $e^{2\operatorname{Re}\lambda} = 1/2$ and so $\operatorname{Re}\lambda = -(\ln 2)/2$. Thus, $r(\sigma(S_{(1,2)}(1)) = 1/\sqrt{2} < 1$, which implies that the zero solution of (1.1) is uniformly exponentially stable.

Now, let us consider $(r_1, r_2) = (1 - \frac{1}{2k+1}, 2)$, where $k \geq 0$ is an odd integer. For k large, this represents a small perturbation of the delays. For this value of the delays, it is easy to verify that the equation (1.2) has a solution $\lambda_k = i(k + \frac{1}{2})\pi$. The corresponding eigenfunction is a solution of (1.1) which is periodic of period $2\pi/\lambda_k$. In partiuclar, this implies that $r(\sigma(S_{(1-\frac{1}{2k+1},2)}(1)) \geq 1$. There is a general result which we will mention later which shows that this quantity actually is equal to 1. For this simple case, this fact also is easy to verify directly. In any case, we have demonstrated the following rather surprising result: *Uniform exponential stability can be destroyed by a small change in the delays.*

It is easy to modify the example (1.1) in such a way that, for the delays $(r_1, r_2) = (1, 2)$ the solution $x = 0$ is uniformly exponentially stable and, for the delays $(r_1, r_2) = (1 - \frac{1}{2k+1}, 2)$, there is a solution which becomes unbounded at an exponential rate. It is only necessary to replace the coefficients $(\frac{1}{2}, \frac{1}{2})$ by $(\frac{1}{2}\beta, \frac{1}{2}\beta)$, where $\beta > 1$, but sufficiently close to 1.

We can obtain the same type of results for the case when we consider only small delays since this is a matter of rescaling time.

Example 1.2 (*Small delays in neutral equations can be bad*). Consider the equation

$$\frac{d}{dt}[x(t) + \frac{1}{2}x(t - r_1) + \frac{1}{2}x(t - r_2)] = -\gamma x(t). \tag{1.3}$$

where $\gamma > 0$ is a constant. If we choose initial data $\varphi \in C$, there is a unique solution $x(t, \varphi)$ of (1.3) which is defined for all $t \in \mathbb{R}$. If we define $[T_{(r_1,r_2)}(t)\varphi](\theta) = x(t + \theta, \varphi)$, $\theta \in [-c_0, 0]$, then $T_{(r_1,r_2)}(t)$, $t \geq 0$, is a C^0-semigroup on C.

We make use of the following result (see Hale and Verduyn-Lunel (1993)), which can be proved using the Laplace Transform.

Theorem 1.2.

(i) $r(\sigma(T_{(r_1,r_2)}(1)) = e^{\beta}$, where β is defined by

$$\beta \equiv \beta(r_1, r_2) \equiv \sup\{\operatorname{Re}\lambda : \Delta_{\gamma,(r_1,r_2)}(\lambda) = 0\}, \tag{1.4}$$

$$\Delta_{\gamma,(r_1,r_2)}(\lambda) = \lambda[1 + \frac{1}{2}e^{-\lambda r_1} + \frac{1}{2}e^{-\lambda r_2}] + \gamma.$$

(ii) $r_e(\sigma(T_{(r_1,r_2)}(1)) = r(\sigma(S_{(r_1,r_2)}(1))$, *where* $S_{(r_1,r_2)}(t)$ *is the semigroup for (1.1).*

Let us apply this result for different values of (r_1, r_2). If we choose the delays $(r_1, r_2) = (1,2)$, then, from Example 1.1, we know that $r_e(\sigma(T_{(r_1,r_2)}(1)) = 1/\sqrt{2} < 1$. Also, it is possible to choose γ so that there is a solution λ_0 of $\Delta_{\gamma,(r_1,r_2)}(\lambda) = 0$ with $\operatorname{Re}\lambda_0 > -(\ln 2)/2$. From Theorem 1.2, this means that the exponential decay rate of the semigroup $T_{(1,2)}(t)$ is determined by an element of the point spectrum and not from the essential spectrum.

Now, let us choose $(r_1, r_2) = (1 - \frac{1}{2k+1}, 2)$. From Example 1.1, we know that $r(\sigma(S_{(1-\frac{1}{2k+1},2)}(1)) = 1$. From Theorem 1.2, this implies that $r_e(\sigma(T_{(r_1,r_2)}(1)) = 1$. *The uniform exponential stability of the origin is destroyed by a small change in* (r_1, r_2) *and it has nothing to do with the constant* γ.

As in Example 1.1, we can obtain exponential growth by changing slightly the coefficients in the difference equation.

We can also achieve the same effect with the delays being small if we simply rescale time $t \mapsto t/\epsilon$ and changing the constant $\gamma \mapsto \epsilon\gamma$, where ϵ is a small parameter. The resulting equation for $\epsilon = 0$ is the ordinary differential equation $\dot{x} = -(\gamma/2)x$, which is uniformly exponentially stable. Changes in the delays as above destroy this type of stability.

Why is it to be expected that such drastic changes in the dynamics as exhibited in Examples 1.1 and 1.2 can result from small changes in the delay? General results on continuous dependence of the spectrum of bounded operators on a Banach space assume that the perturbations are bounded . Even the radius of the essential spectrum cannot increase drastically under such perturbations. If we consider the operator $D_{(r_1,r_2)} : C \to C$ defined by

$$D\varphi = \varphi(0) + \frac{1}{2}\varphi(-r_1) + \frac{1}{2}\varphi(-r_2),$$

then it is not bounded as a function of (r_1, r_2).

We now put the conclusions from these examples in a more general context. First, we consider the system of neutral differential difference equations

$$\frac{d}{dt}[x(t) - \Sigma_{j=1}^{M} a_j x(t - r_j)] = \Sigma_{j=0}^{M} b_j x(t - r_j), \tag{1.5}$$

where $r_0 = 1$, $0 < r_j \le c_0, a_j, b_j, j = 1, \ldots, M$ are constants. Without loss of generality, we can assume that $r_1 \le r_2 \le \ldots \le r_M$. Let $r = (r_1, \ldots, r_M)$, $a = (a_1, \ldots, a_M)$, $b = (b_0, b_1, \ldots, b_M)$, and let $T_r(t) \equiv T_{(r,a,b)}(t)$ be the C^0-semigroup on C defined by (1.5). Also, let $S_{(r,a)}(t)$ be the C^0-semigroup on $C_0 \equiv C_{0,(r,a)} = \{\varphi \in C : \varphi(0) - \Sigma_{j=1}^M a_j\varphi(-r_j) = 0\}$ defined by the difference equation

$$x(t) - \Sigma_{j=1}^M a_j x(t - r_j) = 0. \tag{1.6}$$

Theorem 1.2 holds in this more general situation (see Hale and Verduyn-Lunel (1993)).

Theorem 1.3.

(i) $r(\sigma(T_{(r,a,b)}(1))) = e^\beta$, *where* β *is defined by*

$$\beta \equiv \beta(r,a,b) \equiv \sup\{\operatorname{Re}\lambda : \Delta_{(r,a,b)}(\lambda) = 0\}, \tag{1.7}$$

$$\Delta_{(r,a,b)}(\lambda) = \lambda[1 - \Sigma_{j=1}^M a_j e^{-\lambda r_j}] - \Sigma_{j=0}^M b_j e^{-\lambda r_j}.$$

(ii) $r_e(\sigma(S_{(r,a)}(1))) = e^\alpha$, *where* α *is defined by*

$$\alpha \equiv \alpha(r,a) \equiv \sup\{\operatorname{Re}\lambda : \Delta_{(r,a)}(\lambda) = 0\}, \tag{1.8}$$

$$\Delta_{(r,a)}(\lambda) = [1 - \Sigma_{j=1}^M a_j e^{-\lambda r_j}].$$

(iii) $r_e(\sigma(T_{(r,a,b)}(1))) = r(\sigma(S_{(r,a)}(1)))$.

From this result, it is clear that the eigenvalues of the difference equation (1.6); that is, the zeros of $\Delta_{(r,a)}(\lambda) = 0$, play a fundamental role in the discussion of the asymptotic behavior of the solutions of (1.5) when we subject the equation to perturbations in the delays. We now discuss some properties of these eigenvalues and relate them to the semigroup $S_{(r,a)}(1)$. The presentation follows Avellar and Hale (1980) and includes some results obtained previously by Moreno (1973), Henry (1974), Melvin (1974), Hale (1975) and Silkowski (1976).

Theorem 1.4.

(i) $r(\sigma(S_{(r,a)}(1)))$ *is continuous in* a.

(ii) $r(\sigma(S_{(r,a)}(1)))$ *is lower semicontinuous in* r.

(iii) $r(\sigma(S_{(r,a)}(1)))$ *is continuous in* r *if the components of* r *are rationally independent.*

(iv) If the components of r *are rationally independent, then* $r(\sigma(S_{(r,a)}(1))) = e^{\rho_0}$, *where*

$$1 = \Sigma_{j=1}^M |a_j| e^{-\rho_0 r_j}.$$

(v) $\operatorname{Re}\sigma(S_{(r,a)}(1)) \subset [\rho_M, \alpha]$, *where* α *is defined in (1.8) and* ρ_M *satisfies*

$$|a_M| e^{-\rho_M r_M} = 1 + \Sigma_{j=1}^{M-1} |a_j| e^{-\rho_M r_j}.$$

Furthermore, if the components of r are rationally independent, then $[\rho_M, \rho_0]$ is the smallest closed interval containing $\operatorname{Re}\sigma(S_{(r,a)}(1))$.

Corollary 1.1. *Exponential stability of the difference equation (1.6) is preserved under small pertubations in the delays if and only if*

$$\Sigma_{j=1}^{M}|a_j| < 1. \tag{1.9}$$

The proof is an easy consequence of Theorem 1.4. In fact, from Theorem 1.4 (iv), if the components of r are rationally independent, then exponential stability of (1.6) is equivalent to having $\rho_0 < 0$. Also, from the definition of ρ_0, we see that $\rho_0 < 0$ if and only if (1.9) is satisfied. Since the rationals are dense in $\mathbb{R}$, we obtain the result.

Corollary 1.2. *If exponentially stability of (1.5) is to be preserved under small perturbations in the delays, then it is necessary and sufficient that (1.9) is satisfied.*

The proof is a direct consequence of Theorem 1.3 (i), (iii), Corollary 1.1 and Rouche's Theorem.

Example 1.1 (Revisited) In Example 1.1, we considered in some detail the behavior of the zeros of the function

$$1 + \frac{1}{2}e^{-\lambda r_1} + \frac{1}{2}e^{-\lambda r_2}.$$

In this case, it is easy to see that $\rho_0(r) = 0$ for all r. Also, if the components of r are rationally independent, Theorem 1.4 implies that $[\rho_2(r), 0]$ is the smallest closed interval containing $\operatorname{Re}\sigma(S_{(r,a)}(1))$. A simple computation shows that $\rho_2(1,2) = -\ln 2$. Since $\rho_2(r)$ is continuous in r, it follows that the smallest closed interval containing $\operatorname{Re}\sigma(S_{(r,a)}(1))$ is $[-\ln 2, 0]$ for any r close to $(1,2)$ which has rationally independent coefficients.

Example 1.3 (*Delays approaching zero*). Let us consider the equation

$$h(\lambda, c, \epsilon) = 1 - 2ce^{-\lambda\epsilon_1} + c^2 e^{-\lambda\epsilon_2} = 0,$$

where $0 < \epsilon_1 < \epsilon_2$, and investigate the behavior of the solutions as $\epsilon = (\epsilon_1, \epsilon_2) \to 0$. Avellar and Hale (1980) prove the following result: *If* $-2|c| < 1 - c^2 < 2|c|$, *then* $\rho_2(\epsilon) \to -\infty$, $\rho_0(\epsilon) \to +\infty$ *as* $\epsilon \to 0$. From Theorem 1.4, if the components of ϵ are rationally independent, then $[\rho_2(\epsilon), \rho_0(\epsilon)]$ is the smallest closed interval containing $\operatorname{Re}\sigma(S_{(r,a)}(1))$. Thus, if c satisfies the above restrictions then the smallest closed interval containing $\operatorname{Re}\sigma(S_{(r,a)}(1))$ approaches the interval $(-\infty, \infty)$ as $\epsilon \to 0$.

It also is possible to show that, if we choose $|c| < 1$ and $|c|\epsilon_2/2\epsilon_1 = 1$, then the closure of the set $\operatorname{Re}\sigma(S_{(r,a)}(1))$ is equal to the interval $[\rho_2(\epsilon), \rho_0(\epsilon)]$.

On the other hand, if we choose $\epsilon_2 = 2\epsilon_1$ and $|c| < 1$, we see that $h(\lambda, c, \epsilon) = 0$ if and only if $\operatorname{Re}\lambda = (1/\epsilon_1)\ln|c|$, which approaches $-\infty$ as $\epsilon_1 \to 0$. These remarks show that, if we change the slope of the line along which $\epsilon \to 0$, then the structure of $\operatorname{Re}\sigma(S_{(r,a)}(1))$ changes drastically.

In the applications, it is not always the case that the delays vary in an independent manner. To be precise, we need some notation. Let $\gamma_j = (\gamma_{j1}, \ldots, \gamma_{jM})$, γ_{jk} nonnegative integers for all $j = 1, \ldots, N$, $k = 1, \ldots, M$, $\gamma_0 = (0, \ldots, 0) \in (\mathbb{R}^+)^M$, $\gamma_j \cdot r = \Sigma_{k=1}^M \gamma_{jk} r_k$, $a = (a_1, \ldots, a_N) \in \mathbb{R}^N$, $b = (b_1, \ldots, b_N) \in \mathbb{R}^N$, and assume that

$$0 < \gamma_1 \cdot r < \gamma_2 \cdot r < \ldots < \gamma_N \cdot r. \tag{1.10}$$

We consider the NDDE

$$\frac{d}{dt}[y(t) + \Sigma_{j=1}^N a_j y(t - \gamma_j \cdot r)] = \Sigma_{j=0}^N b_j y(t - \gamma_j \cdot r). \tag{1.11}$$

The characteristic equation for (1.11) is

$$\Delta_{(r,a,b)}(\lambda) \equiv \lambda \Delta_{(r,a)}(\lambda) + g(\lambda, b, r) = 0, \tag{1.12}$$

where

$$\Delta_{(r,a)}(\lambda) = 1 + \Sigma_{j=1}^N a_j e^{-\lambda \gamma_j \cdot r}, \tag{1.13}$$

$$g(\lambda, b, r) = \Sigma_{j=0}^N b_j e^{-\lambda \gamma_j \cdot r}. \tag{1.14}$$

Along with (1.12), we consider also the difference equation

$$y(t) + \Sigma_{j=1}^N a_j y(t - \gamma_j \cdot r) = 0. \tag{1.15}$$

As before, let $T_{(r,a,b)}(t)$ be the C^0-semigroup on C defined by (1.11) and let $S_{(r,a)}(t)$ be the C^0-semigroup on $C_0 \equiv C_{0,(r,a)} = \{\varphi \in C : \varphi(0) + \Sigma_{j=1}^N a_j \varphi(-\gamma_j \cdot r_j) = 0\}$ defined by (1.15).

For (1.11) and (1.15), the Theorem 1.3 is valid as well as Theorem 1.4, Parts (i), (ii) and (iii). The other statements in Theorem 1.4 are not true unless $N = M$ since the delays are not varying in an independent way. The computation of $r(\sigma(S_{(r,a)}(1)))$ in the general case is much more difficult. To understand how to find this, we introduce some additional notation.

Let

$$Z(a, r) = \{\operatorname{Re}\lambda : \Delta_{(r,a)}(\lambda) = 0\}, \tag{1.16}$$

and $\bar{Z}(a, r) = \operatorname{Cl} Z(a, r)$, the closure of $Z(a, r)$.

We first remark that, if the equation $\Delta_{(r,a)}(\mu+i\nu)=0$ for some real numbers μ,ν, then we must have, for $0\le j\le N$,

$$|a_j|e^{-\mu\gamma_j\cdot r}\le \Sigma_{k\neq j}^N|a_k|e^{-\mu\gamma_k\cdot r}. \tag{1.17}$$

We define the numbers $\rho_j=\rho_j(a,r)$, $0\le j\le N$, if they exist, by the relations

$$|a_j|e^{-\rho_j\gamma_j\cdot r}=\Sigma_{k\neq j}^N|a_k|e^{-\rho_j\gamma_k\cdot r}, \tag{1.18}$$

where $a_0=1$. It is easy to verify that ρ_N and ρ_0 are uniquely defined and have the property that $\rho_N=\rho_0$ if $N=1$ and $\rho_N<\rho_0$ if $N\ge 2$. Furthermore, both $\rho_N(a,r)$ and $\rho_0(a,r)$ are continuous in a,r. Also, from (1.17), it is clear that we have the following relation:

$$\bar{Z}(a,r)\subset[\rho_N(a,r),\rho_0(a,r)]. \tag{1.19}$$

Relation (1.19) gives an estimate on $r(\sigma(S_{(r,a)}(1)))$, but it may be very inaccurate.

The following result gives a a more explicit characterization of the set $\bar{Z}(a,r)$. The proof is an easy consequence of Kronecker's Theorem.

Theorem 1.5. *If $\theta=(\theta_1,\dots,\theta_M)$, $0\le\theta_j\le 2\pi$, $j=1,2,\dots,M$, and*

$$H(\rho,\theta,a,r)=1+\Sigma_{k=1}^N a_k e^{-\rho\gamma_k\cdot r}e^{i\gamma_k\cdot\theta}, \tag{1.20}$$

and the components of r are rationally independent, then $\rho\in\bar{Z}(a,r)$ if and only if there is a θ such that $H(\rho,\theta,a,r)=0$.

Corollary 2.1. *For any $r\in(\mathbb{R}_0^+)^M$, $\bar{Z}(a,r)$ is the union of a finite number of intervals.*

Proof. If the components of r are rationally independent, then $\bar{Z}(a,r)$ is characterized by the solutions of $H(\rho,\theta,a,r)=0$, where H is defined in (1.20). Since these solutions are analytic varieties, it is impossible to have the following property: there exists a $\rho\in\bar{Z}(a,r)$, $\{\rho_j\}\in\bar{Z}(a,r)$, $\rho_j\to\rho$ as $j\to\infty$, $(\rho_{j+1},\rho_j)\cap\bar{Z}(a,r)=\emptyset$. This proves the result when the components of r are rationally independent.

For any $r\in(\mathbb{R}_0^+)^M$, there exists a $\beta\in(\mathbb{R}_0^+)^q$ for some integer q such that the components of β are rationally independent. Apply the previous result to β to complete the proof.

Example 1.4. Let us consider the characteristic equation

$$\Delta_0(\mu,h)=1+ke^{-\mu h}-ke^{-2a}e^{-\mu(2+h)}+e^{-2a}e^{-2\mu}, \tag{1.21}$$

where $a \geq 0, k > 0$ are constants. From Theorem 1.5, $\rho \in \bar{Z}(h) \equiv \mathrm{Cl}\,\{\mathrm{Re}\,\mu : \Delta_0(\mu, h) = 0\}$ if an only if there exist $\theta_1, \theta_2 \in [0, 2\pi]$ such that

$$e^{2i\theta_2} = e^{2(\rho+a)} \frac{1 + ke^{-\rho h} e^{i\theta_1}}{-1 + ke^{-\rho h} e^{i\theta_1}}. \tag{1.22}$$

If we define

$$k_0(\rho) = \frac{1 - e^{-2(\rho+a)}}{1 + e^{-2(\rho+a)}}, \tag{1.23}$$

and, if there is a ρ such that

$$ke^{-\rho h} = k_0(\rho),$$

then the right hand side of (1.22) has modulus 1, which implies that $\rho \in \bar{Z}(h)$. As a consequence, for any $\rho > 0$, if $k > k_0(\rho)$, then we can find an $h = h(\rho, k) > 0$ such that $ke^{-\rho h} = k_0(\rho)$; that is, we have $\rho \in \bar{Z}(h)$ and the radius of the essential spectrum of the semigroup at $t = 1$ is larger than 1. If $k > 1$, then this implies that we can choose a ρ as large as we want and then choose an h sufficiently small so that $\rho \in \bar{Z}(h)$. If $k > k_0(0)$, then there exist small ρ and small h such that $\rho \in \bar{Z}(h)$.

If we take the delay $h = 0$ in (1.21), then the solutions of (1.21) must have negative real parts for any $k > 0$; that is, we have expontential stability for the semigroup of the corresponding difference equation. On the other hand, if $k > 1$, we can choose a ρ as large as we want and then choose an h sufficiently small so that $\rho \in \bar{Z}(h)$; that is, there is a solution of the difference equation becoming unbounded at a very large exponential rate.

In the next section, we show that this example occurs also in the characteristic equation arising from the control of a wave equation with delayed boundary control.

For other examples, see Avellar and Hale (1980), where they also discuss similar problems for matrix equations. Further discussion is contained in Hale and Verduyn-Lunel (1993).

2. Delayed Boundary Control. In recent years, there has been considerable effort devoted to the problem of stabilization and control of PDE through the application of forces on the boundary. The mathematical theory is very complete in the situation when the boundary forces are applied with no delays in time (see, for example, Lions (1988)). It has been pointed out recently that a small time delay in the application of the boundary forces can lead to a destabilization of the system (see Datko, Lagnese and Polis (1986), Datko (1988), (1991), (1994), Desch and Wheeler (1989), Hannsgen, Renardy and Wheeler (1988), Logemann, Rebarber and Weiss (1993), and the references therein). In this section, we give such an example for the wave equation, showing that the drastic changes in the stability properties occur essentially for the same reason that we have noticed in the previous section for difference equations.

Consider the linear wave equation

$$w_{tt} + 2aw_t - w_{xx} + bw = 0, \quad 0 < x < 1, \quad t > 0, \tag{2.1}$$

with the boundary conditions

$$w(0,t) = 0, \quad w_x(1,t) = -kw_t(1, t-h), \tag{2.2}$$

where $a \geq 0, k > 0, h \geq 0, b$ are constants. System (2.1), (2.2) corresponds to a boundary stabilization problem where the control function is $kw_t(1, t-h)$. It is not difficult to show that this system generates a C^0-semigroup $\tilde{S}_{h,k}(t)$ on the space $H^1(0,1) \times L^2(0,1)$. We want to determine how $r_e(\tilde{S}_{h,k}(1))$ depends upon k, h. If we replace $w(x,t)$ by $e^{-at}w(x,t)$ and use the fact that, for any constant c, a term cw corresponds to a compact perturbation of the differential operator w_{xx}, then we see that $r_e(\tilde{S}_{h,k}(1)) = e^{-a} r_e(S_{h,k}(1))$, where $S_{h,k}(t)$ is the semigroup generated by the equation

$$w_{tt} - w_{xx} = 0, \quad 0 < x < 1, \quad t > 0, \tag{2.3}$$

with the boundary conditions

$$w(0,t) = 0, \quad w_x(1,t) = -ke^{ah}[-aw(1,t-h) + w_t(1,t-h)]. \tag{2.4}$$

These equations can be written in an equivalent form

$$u_t = -v_x, \quad v_t = -u_x, \tag{2.5}$$

with the boundary conditions

$$v(0,t) = 0, \quad u_t(1,t) = -ke^{ah}[-av(1,t-h) + v_t(1,t-h)]. \tag{2.6}$$

Let us show that (2.5), (2.6) is equivalent to a NDDE with three delays. The general solution of the partial differential equation is

$$v(x,t) = \varphi(x-t) + \psi(x+t),$$

$$u(x,t) = \varphi(x-t) - \psi(x+t).$$

This implies that

$$2\varphi(x-t) = v(x,t) + u(x,t)$$
$$2\psi(x+t) = v(x,t) - u(x,t).$$

From these expressions, we deduce that

$$
\begin{aligned}
2\varphi(-t) &= v(1,t+1) + u(1,t+1) \\
2\psi(t) &= v(1,t-1) - u(1,t-1).
\end{aligned} \tag{2.7}
$$

Using (2.7) and the first boundary condition at $t-1$, we deduce that

$$u(1,t) - u(1,t-2) = -v(1,t) - v(1,t-2).$$

Differentiating this expression with respect to t, using the second boundary condition and letting $y(t) = v(1,t)$, we conclude that

$$\frac{d}{dt}[y(t) + ke^{ah}y(t-h) + y(t-2) - ke^{ah}y(t-2-h)] = ake^{ah}y(t-h) - ake^{ah}y(t-2-h).$$

From this equation, we see that the essential spectral radius $r_e(S_{h,k}(1))$ of $S_{h,k}(1)$ is determined by the supremum of the real parts of the solutions of the equation

$$1 + ke^{ah}e^{-\lambda h} - ke^{ah}e^{-\lambda(2+h)} + e^{-2\lambda} = 0.$$

From our derivations above, $r_e(\tilde{S}_{h,k}(1))$ of $\tilde{S}_{h,k}(1)$, is determined by the supremum of the real parts of the solutions of the equation (let $\lambda - a = \mu$)

$$1 + ke^{-\mu h} - ke^{-2a}e^{-\mu(2+h)} + e^{-2a}e^{-2\mu} = 0. \tag{2.8}$$

If this supremum is negative (positive), then solutions approach zero exponentially (there are solutions which are exponentially unbounded) as $t \to \infty$.

Equation (2.8) is the same as Equation (1.21) in Example 1.4. As a consequence of our discussion in that example, we conclude that

(i) $h = 0$, $k > 0$ implies exponential approach to zero.

(ii) $h > 0$, $0 < k < (1 - e^{-2a})/(1 + e^{-2a})$ implies exponential approach to zero.

(iii) $k > (1 - e^{-2a})/(1 + e^{-2a})$ implies that there is a dense set of $h > 0$ such that there are solutions which are exponentially unbounded.

(iv) If $k > 1$, then we can choose a ρ as large as we want and then choose an h sufficiently small so that there is a solution of (2.1), (2.2) which is becomes unbounded at the rate $e^{\rho t}$ as $t \to \infty$.

As a consequence of this observation, it follows that the system could have been stabilized with a control which involves no delay and then there are arbitrarily small delays in the control which lead to destabilization. These remarks are contained in Datko (1991), (1994), but his proof is somewhat different.

The fact that the above linear wave equation is equivalent to a NDDE is a consequence of a much more general theory (see, for example, Hale and Verduyn-Lunel (1993) and the references therein).

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